CH 0 – PROLOGUE

❑ *THE REAL NUMBERS*

The term **real number** refers to any number that is a **decimal**, or can be written as a decimal.

Examples of Real Numbers

14	14 is written 14.0
7.45	It already is a decimal.
$-\frac{4}{5}$	Same as –0.8
$0.757575...$	It already is a decimal.
π	3.14159. . .
$-\sqrt{2}$	–1.41421 . . .
$\frac{2}{3}$	0.666 . . .

So, what's NOT a *real number*? The classic example is

$$\sqrt{-1}$$

We ask ourselves: What number, when squared, equals –1? No way! Every number squared is at least 0, and most likely positive, never negative.

We thus have a number, $\sqrt{-1}$, that cannot be written as a decimal, and therefore is not a real number.

But we do call it an **imaginary number** denote it with the letter *i*.

$$i = \sqrt{-1}$$

Homework

1. Explain why each of these numbers is classified as a **real number**:

   a. 34.56 b. -99 c. $\frac{\pi}{3}$ d. $\sqrt{44}$ e. $\sqrt{25}$

   f. $\frac{4}{5}$ g. $\frac{23}{99}$ h. $0.010010001\ldots$

   i. 2.74365 j. $-44.90867633\ldots$ k. 0 l. $\sqrt[3]{-64}$

2. Explain why each of these numbers is not a real number:

   a. $\sqrt{-9}$ b. $\sqrt{-30}$ c. $\sqrt[4]{-16}$

❑ *TWO IMPORTANT THINGS WE DO TO REAL NUMBERS*

Opposite

The ***opposite*** of a real number is found by changing the sign of the number. For example, the opposite of 7 is –7, the opposite of $-\pi$ is π, and the opposite of 0 is 0 (since 0 doesn't really have a sign). The opposite of n is $-n$, and the opposite of $-n$ is n. Also notice that the sum of a number and its opposite is always 0; for example, $17 + (-17) = 0$.

When considering numbers on the real number line, two numbers are opposites of each other if they're the same distance from 0, but on opposite sides of 0. [Note that although 0 is the opposite of 0, it's kind of hard to justify the claim that they're on "opposite" sides of 0.]

Homework

3. What is the **opposite** of each number?

   a. 17 b. 0 c. -3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.

   b. The opposite of 0 is ____.

   c. The opposite of a negative number is always _____.

   d. The opposite of a positive number is always _____.

   e. The sum of a real number and its opposite is always _____.

5. Using the formula $y = -x$, find the y-value for the given x-value:

   a. $x = 9$ b. $x = -3$ c. $x = 0$ d. $x = \pi$ e. $x = -\sqrt{2}$

Reciprocal

The ***reciprocal*** of a real number is found by dividing the number into 1. Equivalently, the reciprocal of x is $\frac{1}{x}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. Every real number has a reciprocal except 0; the reciprocal of 0 would be $\frac{1}{0}$, which is undefined, as explained in detail later in this Prologue.

Notice that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative. In addition, **the product of any real number with its reciprocal is always 1**; for example, $\frac{2}{7} \cdot \frac{7}{2} = 1$.

Homework

6. Find the **reciprocal** of each real number:

a. 5 b. $\frac{2}{9}$ c. $-\frac{7}{3}$ d. 1 e. 0 f. $\frac{1}{\pi}$ g. $-\sqrt{3}$

7. a. T/F: Every number has a reciprocal.

b. The reciprocal of 0 is ____.

c. The reciprocal of a negative number is always _____.

d. The reciprocal of a positive number is always _____.

e. The product of a real number and its reciprocal is always ____.

8. Using the formula $y = \frac{1}{x}$, answer each question:

a. If $x = 14$, then $y =$ ____. b. If $x = \frac{2}{3}$, then $y =$ ____.

c. If $x = -99$, then $y =$ ____. d. If $x = -\frac{5}{4}$, then $y =$ ____.

e. If $x = 0$, then $y =$ ____.

☐ *THE FIVE LAWS OF EXPONENTS*

A. $x^3x^4 = x^7$ B. $\frac{x^{10}}{x^2} = x^8$

C. $(xy)^5 = x^5y^5$ D. $\left(\frac{x}{y}\right)^{10} = \frac{x^{10}}{y^{10}}$

E. $(x^3)^4 = x^{12}$ F. $x^0 = 1$ (as long as $x \neq 0$)

G. $\frac{w^{12}}{w^{15}} = \frac{1}{w^3}$ H. $xy^0 + (xy)^0 = x(1) + 1 = x + 1$

I. a^3b^4 cannot be simplified (the bases are different).

Homework

9. Simplify each expression:

a. x^7x^4 b. $\frac{y^4}{y^3}$ c. $(r^4)^5$ d. $(ab)^{12}$

e. $\left(\frac{p}{q}\right)^4$ f. $a^3a^3a^2$ g. $(abw)^3$ h. $(2h)^0 + 2h^0$

☐ *FRACTIONS*

Operations with Fractions

$$-\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\boldsymbol{\frac{7}{6}}$$

$$\frac{4}{5}-\left(-\frac{2}{3}\right) = \frac{4}{5}+\frac{2}{3} = \frac{12}{15}+\frac{10}{15} = \boldsymbol{\frac{22}{15}}$$

$$\frac{2}{9}-7 = \frac{2}{9}-\frac{63}{9} = -\boldsymbol{\frac{61}{9}}$$

$$\left(\frac{2}{3}\right)\left(-\frac{5}{7}\right) = -\boldsymbol{\frac{10}{21}}$$

$$-\frac{4}{7}\div-2 = -\frac{4}{7}\times-\frac{1}{2} = \frac{4}{14} = \boldsymbol{\frac{2}{7}}$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = \boldsymbol{\frac{1}{120}}$$

$$\frac{-8}{\frac{1}{3}} = -\frac{8}{1}\div\frac{1}{3} = -\frac{8}{1}\times\frac{3}{1} = \boldsymbol{-24}$$

$$\frac{-\frac{9}{4}}{-2} = -\frac{9}{4}\div-2 = -\frac{9}{4}\div-\frac{2}{1} = -\frac{9}{4}\times-\frac{1}{2} = \boldsymbol{\frac{9}{8}}$$

Note: A negative sign can "float." For instance,

$$\frac{-30}{6} = \frac{30}{-6} = -\frac{30}{6}$$

since all of these fractions have the value –5.

Powers and Square Roots of Fractions

An **exponent** still means what it always has, so these next examples should be clear.

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}\cdot\frac{2}{3} = \mathbf{\frac{4}{9}} \qquad \left(-\frac{1}{4}\right)^3 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) = -\mathbf{\frac{1}{64}}$$

$$\left(-\frac{9}{4}\right)^1 = -\mathbf{\frac{9}{4}} \qquad \left(\frac{1}{2}\right)^8 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \mathbf{\frac{1}{256}}$$

As for the **square root sign**, we still ask: What number (that's not negative) times itself gives the number in the radical sign?

$\sqrt{\frac{9}{25}} = \mathbf{\frac{3}{5}}$ This is true because $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

$\sqrt{\frac{1}{144}} = \mathbf{\frac{1}{12}}$ This is due to the fact that $\frac{1}{12}\times\frac{1}{12} = \frac{1}{144}$.

$\sqrt{-\frac{4}{49}}$ does <u>not</u> exist as a real number, because $-\frac{4}{49}$ is a negative number, and square roots of negative numbers are outside the real numbers. It's an imaginary number.

$\sqrt{\frac{-4}{-49}}$ <u>does</u> exist as a real number, because the fraction is actually a positive number: $\sqrt{\frac{-4}{-49}} = \sqrt{\frac{4}{49}} = \mathbf{\frac{2}{7}}$.

Homework

Perform the indicated operation:

10. a. $-\frac{1}{2}-\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $\frac{2}{3}-\left(-\frac{5}{6}\right)$

d. $-\frac{4}{5}+\frac{2}{3}$ e. $9-\frac{4}{5}$ f. $-1-\frac{2}{3}$

g. $\frac{8}{3}-5$ h. $-\frac{2}{3}-(-1)$ i. $-\frac{1}{4}-\frac{2}{7}$

11. a. $\left(-\frac{1}{2}\right)\left(-\frac{5}{6}\right)$ b. $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)$ c. $-\frac{5}{6}\cdot-\frac{6}{5}$

d. $-\frac{2}{3}\div-\frac{3}{2}$ e. $\frac{1}{2}\div-9$ f. $7\div-\frac{3}{4}$

g. $\dfrac{-\frac{2}{3}}{-\frac{1}{9}}$ h. $\dfrac{\frac{4}{5}}{-8}$ i. $\dfrac{-\frac{4}{5}}{\frac{5}{8}}$

12. True/False: $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ [assuming $b \neq 0$]

13. a. $\left(-\frac{2}{3}\right)^2$ b. $\left(-\frac{1}{2}\right)^3$ c. $\left(-\frac{1}{3}\right)^4$

d. $\left(-\frac{14}{19}\right)^1$ e. $\left(-\frac{1}{2}\right)^5$ f. $\left(-\frac{2}{3}\right)^6$

14. a. $\sqrt{\frac{81}{100}}$ b. $\sqrt{\frac{36}{64}}$ c. $\sqrt{\frac{1}{4}}$

d. $\sqrt{\frac{1}{9}}$ e. $\sqrt{\frac{121}{144}}$ f. $\sqrt{-\frac{25}{81}}$

g. $\sqrt{\frac{-256}{-289}}$ h. $\sqrt{-\frac{14}{17}-\left(-\frac{14}{17}\right)}$

❑ ORDER OF OPERATIONS

Order of Operations
Parentheses and Brackets [( )]
Exponents
Multiply & Divide (left to right)
Add & Subtract (left to right)

Note: Certainly $(-5)^2 = 25$, since both the 5 and the minus sign are being squared [i.e., $(-5)^2 = (-5)(-5) = 25$]. However, consider the expression

$$-5^2$$

Do we square the –5? The answer is NO; the exponent attaches to the 5 only. The justification is the Order of Operations, which states that exponents (near the top of the chart) are to be done before we deal with negative signs (which are at the bottom of the chart). So, although $(-5)^2 = 25$, we must agree that

$$-5^2 = -25$$

Homework

15. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

16. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

☐ DIVISION AND THOSE PESKY ZEROS

The result of dividing by zero

It's a mathematical fact of life that the only number that's never allowed to be in the denominator (bottom) of a fraction is zero. Sometimes this is phrased

"Never divide by zero."

What's the big deal?

Recall from elementary school that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

Zero on the Top (but not on the bottom)

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work.

Zero on the Bottom (but not on the top)

Now let's put a zero on the bottom and see what happens:

$$\frac{9}{0} = ???$$

Let's try an answer of 0; unfortunately $0 \cdot 0 = 0$, not 9.

How about we try an answer of 9? Then $9 \cdot 0$ is also 0, not 9.

Could the answer be π? No; $\pi \cdot 0 = 0$, not 9.

In fact, any number we surmise as the answer will have to multiply with 0 to make a product of 9. But this is impossible, since any number times 0 is always 0, never 9. In short, no number in the whole world will work in this problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0; in fact, since $0 \cdot 0 = 0$, a possible answer is 0.

Let's try an answer of 5; because $5 \cdot 0 = 0$, another possible answer is 5.

Could π possibly work? Since $\pi \cdot 0 = 0$, another possible answer is π.

Is there any end to this madness? Apparently not, since any number we conjure up will multiply with 0 to make a product of 0. In short, every number in the whole world will work in this problem.

Summary:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Our final summary:

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

> "Black holes are where God divided by zero."
>
> *Steven Wright*

Homework

17. Evaluate each expression, and explain your conclusion:

a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

18. Evaluate each expression:

a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot 5-2\cdot 10}{3^3-9}$

19. $\frac{0}{\pi} = 0$ because
    a. 0 is the only number multiplied by π that will produce 0.
    b. no number times π equals 0.
    c. every number times π equals 0.

20. $\frac{0}{0}$ is undefined because
    a. no number times 0 equals 0.
    b. every number times 0 equals 0.
    c. any number divided by itself is 1.

21. $\frac{7}{0}$ is undefined because
    a. 0 is the only number multiplied by 0 that will produce 7.
    b. no number times 0 equals 7.
    c. every number times 0 equals 7.

22. a. The numerator of a fraction is 0. What can you conclude?
    b. The denominator of a fraction is 0. What can you conclude?

❑ *LINEAR EQUATIONS AND FORMULAS*

Solve for x: $\mathbf{2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)}$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

$\Rightarrow \quad 6x - 14 - 5 + 15x = 4x - 1 + x + 7$ (distribute)

$\Rightarrow \quad 21x - 19 = 5x + 6$ (combine like terms)

$\Rightarrow \quad 21x \mathbf{- 5x} - 19 = 5x \mathbf{- 5x} + 6$ (subtract $5x$ from each side)

$\Rightarrow \quad 16x - 19 = 6$ (simplify)

$\Rightarrow$ $16x - 19 \mathbf{+ 19} = 6 \mathbf{+ 19}$ (add 19 to each side)

$\Rightarrow$ $16x = 25$ (simplify)

$\Rightarrow$ $\frac{16x}{\mathbf{16}} = \frac{25}{\mathbf{16}}$ (divide each side by 16)

$\Rightarrow$ $\boxed{x = \frac{25}{16}}$ (simplify)

Homework

23. Solve each equation:

a. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

b. $2(-8e - 6) - 8(-3e - 2) = 3(-8e - 7) - 4(-2e + 9)$

c. $5(-9r - 5) + 3(8r + 3) = -2(8r - 3) - 3(7r + 7)$

d. $9(-7j - 6) - 7(-5j + 3) = 6(8j + 1) + 5(5j - 8)$

e. $-6(-9d + 6) + 3(-3d - 9) = -8(-d - 9) - 8(-3d - 4)$

Solve for x: $\boldsymbol{\frac{nx - w}{y + z} = e - f}$

Solution: Notice the use of parentheses in the solution.

$\frac{nx - w}{y + z} = e - f$ (original formula)

$\Rightarrow$ $\frac{nx - w}{y + z}(y + z) = (e - f)(y + z)$ (multiply each side by $y + z$)

$\Rightarrow$ $nx - w = (e - f)(y + z)$ (simplify)

$$\Rightarrow \quad nx = (e-f)(y+z)+w \qquad \text{(add } w \text{ to each side)}$$

$$\Rightarrow \quad \boxed{x = \frac{(e-f)(y+z)+w}{n}} \qquad \text{(divide each side by } n\text{)}$$

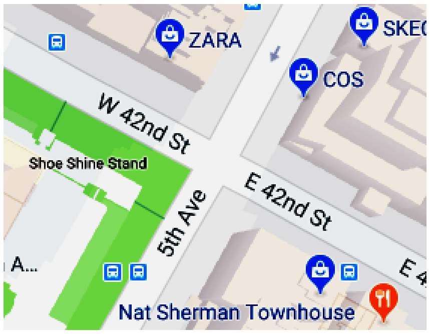
Homework

24. Solve each formula for x:

a. $x - c = d$
b. $2x + b = R$
c. $abx = c$
d. $\dfrac{x}{u} = N$
e. $x(y + z) = a$
f. $\dfrac{x}{n} = c - d$
g. $\dfrac{x}{a+b} = m - n$
h. $\dfrac{x}{c-Q} = c + Q$
i. $\dfrac{x}{R} = a - b + c$
j. $x(b_1 + b_2) = A$
k. $\dfrac{x}{a} - e = m$
l. $\dfrac{x+a}{b} = y$
m. $\dfrac{ax-by}{c} = z$
n. $\dfrac{cx-a}{y+z} = h - g$
o. $\dfrac{ax+b}{c} - d = Q$
p. $\dfrac{9x+u-w}{Q+R} = m + n$
q. $9x - 7y + 13 = 0$

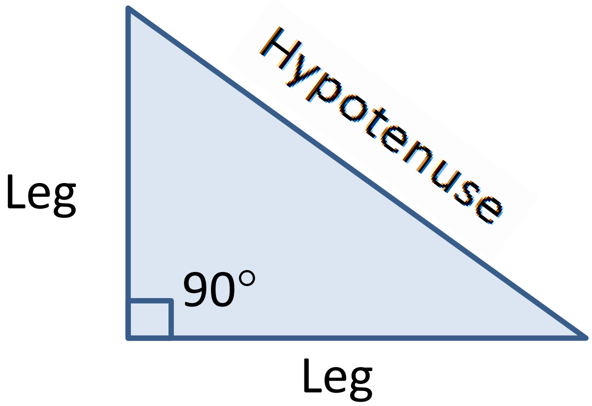
❑ THE PYTHAGOREAN THEOREM

The Right Triangle



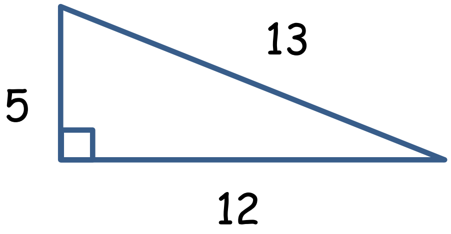
An angle of 90° is called a ***right angle***, and when two things meet at a right angle, we say they are ***perpendicular***. For example, the angle between the floor and the wall is 90°, so the floor is perpendicular to the wall. And in Manhattan, 5th Avenue is perpendicular to 42nd Street.

If a triangle has a 90° angle in it, we call it a ***right triangle***. The two sides that form the right angle (90°) are called the ***legs*** of the right triangle, and the side opposite the right angle is called the ***hypotenuse*** (accent on the 2nd syllable). It also turns out that the hypotenuse is always the longest side of a right triangle.



The Pythagorean Theorem

Ancient civilizations discovered that a triangle with sides 5, 12, and 13 would actually be a right triangle; that is, a triangle with a 90° angle in it.
[By the way, is it obvious that the hypotenuse must be the side of length 13?]



A Classic Right Triangle

But what if just the two legs are known? Is there a way to calculate the length of the hypotenuse? The answer is yes, and the formula dates back to 600 BC, the time of Pythagoras and his faithful followers.

To discover this formula, let's rewrite the three sides of the above triangle:

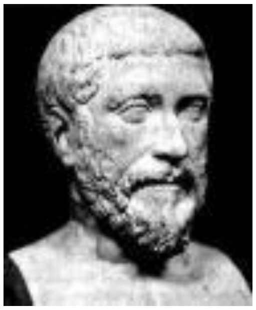
$$leg = 5 \qquad leg = 12 \qquad hypotenuse = 13$$

Here's the secret: Use the idea of <u>squaring</u>. If we square the 5, the 12, and the 13, we get 25, 144, and 169; that is,

$$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$$

and we notice that the <u>sum</u> of 25 and 144 is 169:

$$25 + 144 = 169$$



In other words, a triangle with sides 5, 12, and 13 forms a right triangle precisely because

$$5^2 + 12^2 = 13^2$$

Now let's try to express this relationship in words — it appears that

When you square the legs of a right triangle and add them together, you get the square of the hypotenuse.

As a formula, we can state it this way:

If a and b are the legs of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

We may have named the formula after Pythagoras, but the Babylonians were using the rule 1000 years before Pythagoras was born.

Solving Right Triangles

EXAMPLE: **The legs of a right triangle are 6 and 8. Find the hypotenuse.**

Solution: We begin by writing the Pythagorean Theorem. Then we plug in the known values, and finally determine the hypotenuse of the triangle.

$a^2 + b^2 = c^2$ (the Pythagorean Theorem)

$\mathbf{6}^2 + \mathbf{8}^2 = c^2$ (substitute the known values)

$36 + 64 = c^2$ (square each leg)

$100 = c^2$ (simplify)

What number, when squared, results in 100? A little experimentation yields the solution 10 (since $10^2 = 100$). Also be sure you can use your calculator to calculate $\sqrt{100}$. Our conclusion:

$$\boxed{\text{The hypotenuse is 10}}$$

Note: The equation $100 = c^2$ also has the solution $c = -10$ [since $(-10)^2 = 100$]. But a negative length makes no sense, so we stick with the positive solution, $c = 10$.

Homework

25. In each problem, the two legs of a right triangle are given. Find the **hypotenuse**.

a. 3, 4
b. 5, 12
c. 10, 24
d. 30, 16
e. 7, 24
f. 12, 16
g. 30, 40
h. 9, 40
i. 12, 35
j. 20, 21
k. 48, 55
l. 13, 84
m. 17, 144
n. 11, 60
o. 51, 140
p. 24, 143

❑ *INEQUALITIES*

You must score *between* 80% and 89% to get a B in your math class.

You must be *at least* 18 years of age to vote.

You can be *no taller* than 48 inches to play in the park.

These are all examples of quantities being greater than something or less than something. Since they are not equalities, they are called ***inequalities***.

We know that 5 is bigger than 3, which we can write as "$5 > 3$." The symbol "$>$" can also be read as "is larger than" or "is greater than."

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is less than 5. This is written "$3 < 5$."

The symbol "$\geq$" can be read "is greater than or equal to." For example, $9 \geq 7$ because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than or equal to 7.) And believe it or not, $12 \geq 12$ is a true statement — after all, since $12 = 12$, it's certainly the case that 12 is greater than or equal to 12.

$>$ means "is greater than"

$<$ means "is less than"

$\geq$ means "is greater than or equal to"

$\leq$ means "is less than or equal to"

The symbol "$\leq$" is read "less than or equal to." A couple of examples are $6 \leq 10$ and $8 \leq 8$.

Homework

26. T/F:

a. $7 > 3$

b. $-2 < 1$

c. $13 \geq 13$

d. $-9 \leq -9$

e. $12 \geq 9$

f. $-18 \leq -20$

g. $\pi > 0$

h. $-\sqrt{2} \leq 0$

27. Express each statement as an inequality:

    a. Your age, a, must be at least 18 years.

    b. Your height, h, can be no taller than 48 inches.

    c. Your years of experience, y, must exceed 10 years.

    d. The number of driving tickets, t, must be fewer than 5.

    e. The mean, μ (Greek letter mu), must be at least 75.

    f. The standard deviation, σ (Greek letter sigma), must be no more than 10.

    g. The energy, E, must be greater than 100.

    h. The mass, m, must be less than 3.7.

Solutions

1. a. It's a decimal.

b. It can be written –99.0, a decimal.

c. It's approximately 1.047198, a decimal.

d. It's approximately 6.6332, a decimal.

e. It's 5, which is the decimal 5.0.

f. It equals 0.8, a decimal.

g. It's a (repeating) decimal, 0.232323 . . .

h. It's a (non-repeating) decimal.

i. It's a (terminating) decimal (or it's repeating with zeros).

j. It's a (non-terminating) decimal.

k. $0 = 0.0$, a decimal.

l. It equals –4, which equals –4.0.

2. a. No real number squared is going to be negative.
b. No real number squared is going to be negative.
c. Any real number to the 4th power is going to be 0 or higher, never negative.

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative e. 0

5. a. -9 b. 3 c. 0 d. $-\pi$ e. $\sqrt{2}$

6. a. $\frac{1}{5}$ b. $\frac{9}{2}$ c. $-\frac{3}{7}$ d. 1 e. Undefined f. π g. $\sqrt{3}$

7. a. False; 0 has no reciprocal. b. Undefined c. negative
d. positive e. 1

8. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

9. a. x^{11} b. y c. r^{20} d. $a^{12}b^{12}$
e. $\frac{p^4}{q^4}$ f. a^8 g. $a^3b^3w^3$ h. 3

10. a. $-\frac{13}{10}$ b. 0 c. $\frac{3}{2}$ d. $-\frac{2}{15}$ e. $\frac{41}{5}$
f. $-\frac{5}{3}$ g. $-\frac{7}{3}$ h. $\frac{1}{3}$ i. $-\frac{15}{28}$

11. a. $\frac{5}{12}$ b. -1 c. 1 d. $\frac{4}{9}$ e. $-\frac{1}{18}$
f. $-\frac{28}{3}$ g. 6 h. $-\frac{1}{10}$ i. $-\frac{32}{25}$

12. True

13. a. $\frac{4}{9}$ b. $-\frac{1}{8}$ c. $\frac{1}{81}$ d. $-\frac{14}{19}$
e. $-\frac{1}{32}$ f. $\frac{64}{729}$

14. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{11}{12}$
f. Not a real number g. $\frac{16}{17}$ h. 0

15. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

16. a. $(x+y)^2 = (2+1)^2 = 3^2 = 9$
b. $x^2 + y^2 = 10^2 + 5^2 = 100 + 25 = 125$
c. 63 d. 96 e. 44

17. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, never 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 is 0; thus EVERY number works.

18. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

19. a. **20**. b. **21**. b.

22. a. You can't conclude anything — it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we can conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

23. a. $a = \frac{19}{16}$ b. $e = -\frac{61}{24}$ c. $r = \frac{1}{16}$
d. $j = -\frac{41}{101}$ e. $d = \frac{167}{13}$

24. a. $x = d + c$ b. $x = \frac{R-b}{2}$ c. $x = \frac{c}{ab}$

d. $x = Nu$ e. $x = \frac{a}{y+z}$ f. $x = n(c - d)$

g. $x = (m - n)(a + b)$ h. $x = (c + Q)(c - Q)$ i. $x = R(a - b + c)$

j. $x = \frac{A}{b_1 + b_2}$ k. $x = a(m + e)$ l. $x = by - a$

m. $x = \frac{cz + by}{a}$ n. $x = \frac{(h-g)(y+z)+a}{c}$

o. $x = \frac{c(Q+d)-b}{a}$ p. $x = \frac{(m+n)(Q+R)+w-u}{9}$

q. $x = \frac{7y-13}{9}$

25. a. 5 b. 13 c. 26 d. 34 e. 25 f. 20
g. 50 h. 41 i. 37 j. 29 k. 73 l. 85
m. 145 n. 61 o. 149 p. 145

26. a. T b. T c. T d. T
e. T f. F g. T h. T

27. a. $a \geq 18$ b. $h \leq 48$ c. $y > 10$ d. $t < 5$
e. $\mu \geq 75$ f. $\sigma \leq 10$ g. $E > 100$ h. $m < 3.7$

“The greatest mistake
you can make in life
is to be continually
fearing you will
make one.”

– **Elbert Hubbard (1856–1915)**

"Excellence is never an accident. It is always the
result of high intention, sincere effort, and
intelligent execution. It represents the wise
choice of many alternatives – choice,
not chance, determines your destiny."

— Aristotle

Table of Contents

“Perplexity

is the beginning of knowledge.”

Kahlil Gibran

CH 1 – LOGIC

In everyday use, "Go see *Star Wars* tonight" might be considered a statement. But in logic and math, it's not a statement because a ***statement*** is an assertion that can clearly be classified as either *true* or *false* (even if we don't know which one).

Rodin's *The Thinker*

☐ *STATEMENTS AND IMPLICATIONS*

"*The quadratic formula works for all quadratic equations*" is a ***statement*** (and it's true). Another statement is "*2 + 2 = 5*" (but it's false). Even "*There are 100 billion stars in our galaxy*" is a statement. I have no idea whether it's true or false, but I'm sure that it's either true or false.

Most of the results in mathematics are actually a special hooking-together of two statements; we call these ***implications***. An implication is sometimes called an "***if-then***" statement. If p and q are statements, then an implication can be written

If *p*, then *q*

or, ***p* implies *q***

or, $\boldsymbol{p \Rightarrow q}$

The essence of computer programming lies in the *if-then* statement.

The statement $\boldsymbol{p}$ is called the ***hypothesis*** and the statement $\boldsymbol{q}$ is called the ***conclusion***.

For example, the fact that the square of an odd number is odd (for example, $5^2 = 25$) can be written

<u>If</u> n is odd, <u>then</u> n^2 is odd,

or n odd $\Rightarrow$ n^2 odd.

Homework

1. Is the following a statement? "There are exactly 7 people in New York City with 32,000 hairs on their heads."

2. Is the following a statement? "Walk tall!"

3. Is the following a statement? "Arvis said, 'Walk tall!' "

4. Goldbach stated many years ago, "Every **even** number starting at 4 is the sum of two **primes**." For example, 20 is an even number, and **20 = 17 + 3**, where 17 and 3 are prime. As of today, this conjecture has never been proven to be true or false — mathematicians simply don't know. Is Goldbach's Conjecture a statement?

5. State the hypothesis and conclusion of the statement "If x is any real number, then $x^2 \geq 0$."

6. State the Pythagorean Theorem in the "if-then" form.

7. "All dogs go to heaven" is an implication. Write it in the "if-then" form.

8. Consider the implication, "I'll get an A if I study hard enough." State the hypothesis and conclusion.

9. Convert each implication into the

   "<u>If</u> *hypothesis*, <u>then</u> *conclusion*" form:

   a. Whenever I eat pie, I get sick.
   b. The sidewalks get wet whenever it rains.
   c. I get A's if I study.
   d. The President must be at least 35 years of age.

☐ NEGATION

The ***negation*** of a statement is the "opposite" of the statement, and is denoted with a squiggle, ~, in front of the statement, although there are other notations. For example, let p be the statement "T is a right triangle." Then ~p is read "not p" and is the statement "T is <u>not</u> a right triangle."

A more subtle example: Let q = "All prime numbers are odd." Then ~q = "Not all prime numbers are odd," meaning, "There is at least one prime number which is not odd." [Can you name it?]

The bottom line for negation is that if p is true, then ~p is false — and if p is false, then ~p is true. Can you see that ~(~p) has the same truth value of p? What does the statement "Nobody doesn't like Sara Lee" mean?

p	$\sim p$
True	False
False	True

From the implication $p \Rightarrow q$ and the negation ~, we can create two variations of the implication, described in the following two sections.

Homework

10. Let p be the statement, "The 49ers lost the Super Bowl." Assume that the game cannot end in a tie.
State ~p.

11. Let p be the statement "$x > 0$". State ~p.

12. Let p be the statement, "Every triangle is a polygon." State ~(~p).

13. Simplify: ~(~(~(~p)))

☐ CONVERSE

The ***converse*** of the statement $p \Rightarrow q$ is $q \Rightarrow p$. We interchange the hypothesis and the conclusion. Consider the statement, "If you live in San Francisco, then you live in California." Its converse is "If you live in California, then you live in San Francisco." The original statement is true, but its converse can easily be false.

What's the converse of the statement "If a triangle has three equal sides, then each of its angles is 60°"? It's the statement, "If each angle of a triangle is 60°, then the triangle has three equal sides." In this example, both the statement and its converse are true.

In short, the converse of a true statement may — or may not — be true.

☐ CONTRAPOSITIVE

Consider the true statement above, "If you live in San Francisco, then you live in California." Though the converse of this statement is false, let's take the converse <u>and</u> negate each part. This is called constructing the ***contrapositive***. If we do this "switch and negate" maneuver, we get the statement, "If you don't live in California, then you don't live in San Francisco." This statement makes sense. Do you agree?

Therefore, the contrapositive of the statement $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$. It seems, at least using the above example, that the contrapositive of a true statement is true, and indeed this is always the case.

Statement	$p \Rightarrow q$	
Its Converse	$q \Rightarrow p$	Truth value has no relation to the truth of the original statement.
Its Contrapositive	$\sim q \Rightarrow \sim p$	Truth value must match that of the original statement.

Homework

14. Let p = "If it's dark, then it's scary."

    a. State the converse of p.

    b. State the contrapositive of p.

    c. Which of the two statements must have the same truth value as p?

15. Let q = "If angle A is a right angle, then A = 90°." State the converse. Is the converse true or false?

16. Let r = "If $x = 5$, then $x^2 = 25$." State the converse. Is the converse true or false?

17. State the converse of the Pythagorean Theorem. Is the converse true or false?

18. Consider the true statement from calculus: "If f is differentiable at x, then f is continuous at x." State the converse and the contrapositive of the statement. Which of the two statements must be true?

19. Consider the false statement from a branch of math called topology: "If $\varnothing$ is in T, then T is a topology." State the converse and the contrapositive of the statement. Which of the two statements must be false?

20. Consider the implication $r \Rightarrow s$. State the converse and the contrapositive of the implication. Which of the two statements must have the same truth value as the original implication?

❑ MORE ON COMBINING STATEMENTS

AND Another way to hook two statements together to make a new statement is to place the word *AND* between them. The meaning is the same as that in English. If both p and q are true, then the statement p AND q is true. But if either p or q (or both) is false, then the statement p AND q is false.

p	q	p AND q
True	True	True
True	False	False
False	True	False
False	False	False

The statement “2 + 2 = 4 AND The letter *a* is a vowel” is true.

The statement “2 + 2 = 4 AND The letter *m* is a vowel” is false.

The statement “2 + 2 = 5 AND The letter *z* is a vowel” is false.

OR Two statements can be linked by the word *OR* to create a new statement. In this case, the logical meaning of OR may not be consistent with the use of the word OR in English. In math, as long as <u>at least one</u> of the statements p or q is true, then the statement p OR q is true. In particular, if both p and q are true, then p OR q is true.

p	q	p OR q
True	True	True
True	False	True
False	True	True
False	False	False

Thus, the following are true statements (since at least one part is true):

2 + 2 = 4 OR Lincoln is our current president.

2 + 2 = 4 OR California is in the U.S.A.

But the following statement is false (since both parts are false):

The earth is cubical OR 10 + 10 = 29.

Note that our use of the word OR is <u>not</u> the way your mother meant the word OR when she said, "You can have ice cream OR a cookie."

The OR we use in logic and math – one or the other or both – is called the **inclusive OR**.

The OR that means one or the other – but <u>not</u> both – is called the **exclusive OR**.

Homework

21. a. What are the conditions for *p* and *q* that will make the statement *p* AND *q* true?

    b. What will make *p* AND *q* false?

22. a. What are the conditions for *p* and *q* that will make the statement *p* OR *q* true?

    b. What will make *p* OR *q* false?

23. T/F: a. "2 + 2 = 4 OR Lincoln was the 1st president."
    b. "2 + 2 = 4 OR $5^2 = 25$.
    c. "1 + 3 = 10 OR I'm 10 feet tall."

24. T/F: a. "2 + 2 = 4 AND Lincoln was the 1st president."
    b. "2 + 2 = 4 AND Washington was the 1st president."
    c. "California is in Idaho AND a circle is a square."

25. "$6 \leq 6$" is a true statement. Explain.

26. When is the statement *p* AND *q* AND *r* true?

27. When is the statement *p* OR *q* OR *r* true?

28. [Optional] Consider the logical connective, denoted by *XOR*, and defined by the truth table:

p	q	p XOR q
True	True	False
True	False	True
False	True	True
False	False	False

Give an example from real life that uses this connective.

❑ *DEFINITIONS, AXIOMS, AND THEOREMS*

There are really only two kinds of statements in all of mathematics: those you don't prove (the definitions and axioms), and those you do prove (the theorems).

Consider the following ***definition***: A *right angle* is an angle whose measure is 90°. [The Babylonians adored the number 360, and 90 is one of its many factors.] Is this definition right or wrong? It's not right or wrong — it just is. If you disagree with the definition, then you can either accept it so you'll pass your math course, or you can define it differently and perhaps create a new branch of mathematics!

An example of an ***axiom*** is the *commutative property* for addition of two numbers: $a + b = b + a$. We accept this as fact — we don't consider whether or not there's a proof. Why do we accept this axiom? Well, it seems to work for all the number systems we've seen in algebra — it seems "intuitively" clear. [Not necessarily the case in advanced math courses.]

A ***theorem***, on the other hand, is a statement that requires a **proof**. The proof of a theorem is based on definitions, axioms, and previously proved theorems. The *Quadratic Formula* is an example of a theorem, and can be expressed as an implication:

If $ax^2 + bx + c = 0$ (where $a \neq 0$), **then** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The classic theorem is the *Pythagorean Theorem*:

> **If** a and b are the legs of a right triangle and c is the hypotenuse, **then** $a^2 + b^2 = c^2$.

At the end of a proof of a theorem we sometimes write Q.E.D., which is Latin for *quod erat demonstrandum*, which loosely translates to "as was to be proved."

Homework

29. A *group* is a semigroup with identity. Is this definition right or wrong?

30. Prove the axiom: If a and b are numbers, then $ab = ba$.

31. The formula for *slope* ($m = \frac{\Delta y}{\Delta x}$) is not a theorem. What is it, then?

32. a. Why is the Distance Formula in the plane considered a theorem?
    b. State the hypothesis and conclusion.
    c. What geometry theorem is the Distance Formula based upon?

33. State the hypothesis and conclusion of the Quadratic Formula.

❑ *EXAMPLES AND COUNTEREXAMPLES*

Consider the statement: If x is any number, then $x^2x^3 = x^5$. Here's an ***example*** of this.

Let $x = 2$. Then,

x^2x^3 gives $2^2 \cdot 2^3 = 4 \cdot 8 = 32$; and

x^5 gives $2^5 = 32$. ✓

What have we proved? Not much — just that $2^2 \cdot 2^3 = 2^5$. But this in no way proves the more general statement, $x^2x^3 = x^5$. It is just an example. It gives some credence to the general statement, but nothing more.

Now consider the statement: If a and b are numbers, then $(a + b)^2 = a^2 + b^2$. Though this statement is true sometimes, it's considered to be a false statement. To prove that a statement is false, all we need is one ***counterexample***: an example that runs counter (contrary) to the statement.

Let $a = 3$ and $b = 4$. Then,

$$(3 + 4)^2 = 7^2 = 49,$$

whereas $3^2 + 4^2 = 9 + 16 = 25.$ ☹

Therefore, $(a + b)^2 = a^2 + b^2$ is a false statement, and it took only one counterexample to prove that it's false.

Homework

34. Consider the statement "$x \ge 0$."
    a. Describe the statement in words.
    b. Give two different x values that make the statement true, and don't make both positive.

35. Consider the statement: $x - y = y - x$ for any real numbers. Is the statement true or false? Prove your answer.

36. Describe precisely when $(a + b)^2 = a^2 + b^2$ is a true statement.

37. A natural number with precisely two factors is called a *prime number*. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, . . .

    a. T/F: Every prime number is odd. Prove your answer.

    b. T/F: Every odd number is prime. Prove your answer.

38. Disprove the statement: $x^2 = 2x$ for any real number x. Hint: Don't let $x = 0$ or 2. Explain why I'm not allowing you to use 0 or 2.

Review Problems

39. a. Give an example of a true statement.
    b. Give an example of a false statement.
    c. Give an example of a statement whose truth is unknown.
    d. Give an example of a sentence which is not a statement.

40. What distinguishes a statement from a non-statement?

41. A *semigroup* is defined to be a set with an associative binary operation. Prove this statement.

42. Simplify the logic expression $\sim(\sim(\sim p))$.

43. A true theorem from group theory states: If H is a subgroup of G, then the order of H divides the order of G.
    a. State the hypothesis of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b must be true?

44. Let p = "2 + 3 = 5" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false?
    b. Is p AND q true or false?

45. a. A "lemma" is a math statement similar to a theorem. Is a lemma a statement we do, or do not, prove?
    b. A "postulate" is a math statement similar to an axiom. Is a postulate a statement we do, or do not, prove?

46. Consider the statement $(x+y)^3 = x^3 + y^3$ for any real numbers x and y.
    a. Give an example to give credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

47. Explain why "Ultraviolet waves have a higher frequency than infrared waves" is a *statement* in logic.

48. One of the axioms of a group is that there must be an identity element. Prove this statement.

49. Simplify the logic expression $\sim(\sim(\sim(\sim(\sim p))))$.

50. A true theorem from group theory states: If H is a subgroup of a cyclic group G, then H is cyclic.
    a. State the conclusion of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b <u>must</u> be true?

51. Let p = "$2 + 3 = 9$" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false? b. Is p AND q true or false?

52. Consider the statement $\sqrt{a^2+b^2} = a+b$ for any real numbers a and b.
    a. Give an example to give some credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

53. True/False
    - a. $p \Rightarrow q$ is called an implication.
    - b. In $p \Rightarrow q$, q is called the hypothesis.
    - c. $\sim(\sim p)$ simplifies to $\sim p$.
    - d. The converse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
    - e. The contrapositive of $\sim q \Rightarrow \sim p$ is $p \Rightarrow q$.
    - f. $x = 2$ is a counterexample to the conjecture $x^2 = 2x$.
    - g. If p is true, q is false, and r is true, then (p OR q) AND r is true.
    - h. Axioms and theorems are proved.
    - i. Definitions are never proved.
    - j. The converse of a true implication is always true.
    - k. The contrapositive of a false implication is always false.

Solutions

1. Yes — I don't know whether it's true or false, but it's certainly one or the other.

2. No — It's simply a command. There's no possible way the sentence can be evaluated as true or false.

3. Yes, this sentence is a statement. Even though it sounds like the previous problem, it makes the statement that Arvis said something. Either he did or he didn't, so the statement is either true or false.

4. Yes — as we're learning, a statement is a sentence with a truth value of either true or false, regardless of whether we mortals know which it is. By the way, 100 is the sum of what two primes?

5. Hypothesis: If x is any real number — Conclusion: $x^2 \geq 0$.

6. If a and b are the legs of a right triangle and c is the hypotenuse, then

$a^2 + b^2 = c^2$.

7. If you're a dog, then you'll go to heaven.

8. Hypothesis: If I study hard enough — Conclusion: I'll get an A

9. a. If I eat pie, then I get sick.

b. If it rains, then the sidewalks get wet.

c. If I study, then I get A's.

d. If one is the President, then one is at least 35 years of age.

10. The 49ers won the Super Bowl.

11. Hint: The answer is <u>not</u> "$x < 0$."

12. Every triangle is a polygon.

13. p

14. a. If it's scary, then it's dark b. If it's not scary, then it's not dark.

c. The contrapositive always has the same truth value as the original statement.

15. If A = 90°, then A is a right angle. The converse is true.

16. If $x^2 = 25$, then $x = 5$. The converse is false (since $x = \pm 5$).

17. If a, b, and c are the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle. The converse is true.

18. Converse: If f is continuous at x, then f is differentiable at x.

Contrapositive: If f is not continuous at x, then f is not differentiable at x.

The contrapositive must be true if the original statement is true.

19. Converse: If T is a topology, then $\varnothing$ is in T.

Contrapositive: If T is not a topology, then $\varnothing$ is not in T.

The contrapositive will be false.

20. Converse: $s \Rightarrow r$ Contrapositive: $\sim s \Rightarrow \sim r$

The contrapositive always has the same truth value as the original statement.

21. a. Both p and q must be true.

b. If either or both of them are false.

22. a. Either p is true, or q is true, or both are true.

b. Only if both of them are false.

23. a. T b. T c. F

24. a. F b. T d. F

25. The statement says, "6 is less than OR equal to 6." Since 6 is equal to 6, and since an OR statement is true as long as one of the parts is true, the entire statement is true.

26. Only when all three parts are true.

27. As long as any one of them (or two of them or all of them) is true.

28. I'm curious what you think the truth table really means.

29. Neither — it's a definition.

30. Axioms aren't proven — they're accepted.

31. It's a definition. (It's not a theorem because it cannot be proven.)

32. a. Because it can be derived (proved).

b. Hypothesis: If (a, b) and (c, d) are points in the plane . . .
Conclusion: then the distance between them is $\sqrt{(a-c)^2+(b-d)^2}$

c. The Pythagorean Theorem

33. Hypothesis: Given the quadratic equation $ax^2 + bx + c = 0$ . . .

Conclusion: the solutions are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

34. a. "x is greater than or equal to zero."

b. If $x = 7$, the statement is true, and if $x = 0$, the statement is true.

35. False — to prove this, we take the *counterexample*: $7 - 2 = 5$, whereas $2 - 7 = -5$.

36. It has something to do with zero.

37. a. See the 2? b. Is 15 prime?

38. Let $x = 5$ and the statement fails. Choosing x as 0 or 2 is a lousy choice because the statement actually works for these numbers.

39. a. For example, "The *Harry Potter* books were written by a woman."

b. For example, "Every parabola has two x-intercepts."

c. For example, "It took exactly 197 geeks to write Windows."

d. For example, "Why did you break the window?"

40. A statement can be assigned a true/false value.

41. A definition cannot be proved.

42. ~ p

43. a. H is a subgroup of G.

b. *converse*: If the order of H divides the order of G, then H is a subgroup of G.

contrapositive: If the order of H does not divide the order of G, then H is not a subgroup of G.

c. The contrapositive must be true.

44. a. True b. False **45**. a. Prove b. Do not prove

46. a. Let $x = 1$ and $y = 0$ and the statement is true. b. One counterexample is to let $x = 2$ and $y = 3$.

47. It's a statement because it's an assertion that is either true or false.

48. It can't be proved; it's an axiom, which we assume to be true.

49. ~ p

50. a. H is cyclic.

b. *converse*: If H is cyclic, then H is a subgroup of a cyclic group G.

contrapositive: If H is not cyclic, then H is not a subgroup of a cyclic group G.

c. The contrapositive must be true.

51. a. false b. false

52. a. Letting $a = 5$ and $b = 0$, for instance, will make the statement true.

b. Letting $a = 4$ and $b = 3$ will make the statement false. We used a *counterexample* to prove the statement false.

53. a. T b. F c. F d. F e. T f. F
g. T h. F i. T j. F k. T

> "The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth."

Niels Bohr, physicist (1885–1962)

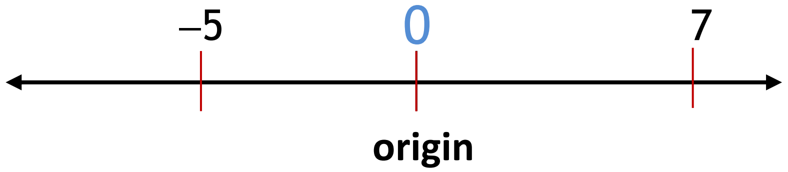
CH XX – ABSOLUTE VALUE, THE BASICS

Sometimes the "size" of a number is more important than the "sign" of the number, that is, whether it's positive, zero, or negative. To measure the size of a number (without regard to its sign), we consider the ***absolute value*** of that number. There are two equivalent approaches to the meaning of absolute value.

☐ TWO WAYS OF LOOKING AT ABSOLUTE VALUE

1) **Geometry**

On a number line, the ***absolute value*** of a number (any kind of real number) is the ***distance*** from that number to **0** (the origin) on the line.



What is the distance from the number **7** to 0? It's 7. So we say that the absolute value of 7 is **7**.

How far from the origin is the number **–5**? It's 5 units away, and thus the absolute value of –5 is **5**.

What is the distance from the origin to **2π**? The distance is 2π, and we conclude that the absolute value of 2π is **2π**.

What is the distance between the number **0** and the origin? It's 0; conclusion: the absolute value of 0 is **0**.

Notes on Distance:

First, we can calculate the distance from any number to 0 (the origin). In other words, the distance between the origin and any point on the line always exists, even if it's hard or impossible to calculate.

Second, distance must be greater than or equal to 0; that is, it's never negative. If $\boldsymbol{d}$ represents distance, then $\boldsymbol{d} \geq \mathbf{0}$.

2) Arithmetic

If a number is positive, the *absolute value* of the number is the same number. For example, the absolute value of 9 is **9**.

If a number is negative, the *absolute value* of the number is the opposite of the number. For example, the absolute value of –3 is **3**.

And the *absolute value* of 0 is simply **0**.

Put more simply, the absolute value of a number greater than or equal to 0 is the same number; if the number is negative, just remove the minus sign to calculate its absolute value.

❑ *NOTATION*

Instead of writing the words *absolute value* all the time, we do what we always do in math: condense the concept into symbols. To represent the absolute

In computers, the *absolute value* of *n* would be written

abs(*n*)

value of a number, we put a vertical bar on each side of the number. Thus,

The absolute value of 9 is 9: $|9| = \mathbf{9}$

The absolute value of –3 is 3: $|-3| = 3$

The absolute value of 0 is 0: $|0| = \mathbf{0}$

Just for repetition: If a number is greater than or equal to 0, then its absolute value is that <u>same</u> number. If a number is less than 0 (which means it's a negative number), then its absolute value is the <u>opposite</u> of that number (which will then turn into a positive number). We can write this in the following way; it looks confusing, but it's totally valid:

If $x \ge 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

This definition ensures that the absolute value of a quantity is never negative.

Here are some more examples of absolute value:

$|17.5| = 17.5$ $\quad |0| = 0$ $\quad |-13| = 13$

$|\pi| = \pi$ $\quad |-\sqrt{7}| = \sqrt{7}$ $\quad |-3\pi| = 3\pi$

Bottom Line:

The ***absolute value*** of a quantity is either positive or zero:

If x is ANY quantity, then $|x| \ge 0$.

In other words, the absolute value of a quantity is <u>never</u> negative.

For example, consider the following expression:

$$\left|\sin^2(\pi/6) - \ln(e-1)\right|$$

Even though you may not be able to calculate it until Pre-calculus, you should still be able to understand that the answer to this problem — whatever it is — is greater than or equal to zero. Equivalently, the answer is not negative.

EXAMPLE 1: **Evaluate each expression:**

A. $|7-2| = |5| = \mathbf{5}$

B. $|3^2-15| = |9-15| = |-6| = \mathbf{6}$

C. $-|2-7| = -|-5| = \mathbf{-5}$

D. $|-3-4|-|10-2| = |-7|-|8| = 7-8 = \mathbf{-1}$

Homework

1. The absolute value of any number is ______.

2. If x represents any number, then

   a. $|x|>0$ b. $|x|<0$ c. $|x|\ge 0$ d. $|x|\le 0$

3. Which one of the following inequalities has NO solution?

   a. $|x|>0$ b. $|x|<0$ c. $|x|\ge 0$ d. $|x|\le 0$

4. True/False:

   a. Every number has an absolute value.
   b. There is a number whose absolute value is 0.
   c. There is a number whose absolute value is negative.
   d. There are two different numbers whose absolute value is 9.

5. Simplify each expression:

   a. $|4-14|$ b. $|2(-3)-7|$ c. $-|-9|$ d. $|2^0-\pi^0|$

6. Simplify each expression:

   a. $|7\pi|$ b. $|-7\pi|$ c. $|\sqrt{67}|$ d. $|-\sqrt{2.7}|$ e. $|0^{3\pi}|$

7. What's the best statement you can make regarding the expression

$$\left|\frac{\tan x - \sin x}{e^{\int \sec x\, dx}}\right| ?$$

8. Consider the statement: $|a \cdot b| = |a| \cdot |b|$

   a. Give three examples where the statement is true, using

      i) two positive numbers.
      ii) two negative numbers.
      iii) two numbers of opposite sign.

   b. Do you think the statement is true or false?

9. Consider the statement: $|x+y| = |x| + |y|$

   a. Give an example where the statement is true.

   b. Give an example where the statement is false.

   c. So, is the statement true or false?

10. Consider the statement: $|x+y| < |x| + |y|$

    a. Give an example where the statement is true.

    b. Give an example where the statement is false.

    c. Is the statement true or false?

11. [Hard] Try to use the previous two problems to come up with a statement that is TRUE.

6

Absolute Value – The Basics

The *absolute value* of a positive number is *itself*.

The *absolute value* of 0 is 0.

The *absolute value* of a negative number is its *opposite*.

Solutions

1. "greater than or equal to 0"
OR "never negative"
OR " ≥ 0 "

2. c.

3. b.

4. a. T b. T c. F d. T

5. a. 10 b. 13 c. -9 d. 0

6. a. 7π b. 7π c. $\sqrt{67}$ d. $\sqrt{2.7}$ e. 0

7. If you don't know enough algebra and trig, the best you could say is that, whatever legal value of *x* you place in the expression, the result WON'T BE NEGATIVE. That is, the expression represents something that is GREATER THAN OR EQUAL TO 0.

8. Although a whole slew of examples does not prove that the statement is true, this particular statement <u>is</u> always true.

9. Recall that if a statement fails in just one case, it's considered a false statement.

10. Again, if a statement fails in just one case, it's considered a false statement.

11. I will not give the answer away, but it's called the *Triangle Inequality*, and is one of the most important rules in math.

*"Who questions much,
shall learn much,
and retain much."*

– Francis Bacon

CH NN – GRAPHING ABSOLUTE VALUES

It's time to do more graphing — of course, this time there will be absolute values in the formula.

EXAMPLE 1: **Graph: $y = |x|$**

Solution: Pick some x-values, calculate the corresponding y-values, and plot the points:

$x = -2 \Rightarrow y = |-2| = 2 \Rightarrow$ **(–2, 2)**

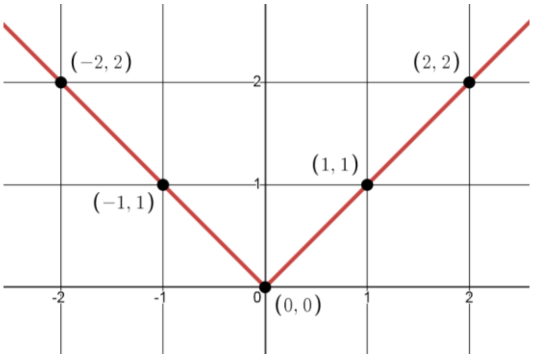
$x = -1 \Rightarrow y = |-1| = 1 \Rightarrow$ **(–1, 1)**

$x = 0 \Rightarrow y = |0| = 0 \Rightarrow$ **(0, 0)**

$x = 1 \Rightarrow y = |1| = 1 \Rightarrow$ **(1, 1)**

$x = 2 \Rightarrow y = |2| = 2 \Rightarrow$ **(2, 2)**

We've got five points, enough to make a decent picture:



$y = |x|$

Convince yourself that our graph also contains the points (165, 165) and (–499, 499).

EXAMPLE 2: **Graph:** $y = |x - 3| + 2$

Solution: Don't panic! Let's just do what we did in the first example: Pick *x*-values off the top of our head, calculate the associated *y*-values, plot the points (*x*, *y*), and see what we get.

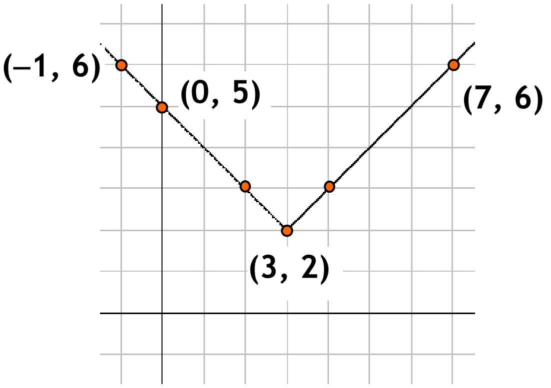
Setting *x* to 0 gives $y = |0-3|+2 = |-3|+2 = 3+2 = \mathbf{5}$.
We now have our first point on the graph: (0, 5).

Now set *x* to 7: $y = |7-3|+2 = |4|+2 = 4+2 = \mathbf{6}$.
Our new point on the graph is (7, 6).

One more — set *x* to –1: $y = |-1-3|+2 = |-4|+2 = 4+2 = \mathbf{6}$.
Our third point is (–1, 6).

We'll now construct an *x*-*y* table using the three points we've just calculated and some additional points; it's your job to verify the rest of the points.

x	*y*
–1	6
0	5
1	4
2	3
3	2
4	3
5	4
6	5
7	6



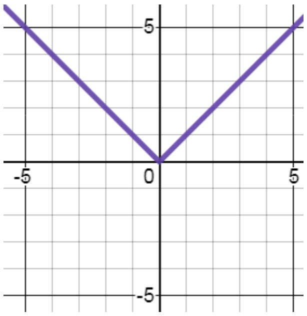
$y = |x - 3| + 2$

Items of Note:

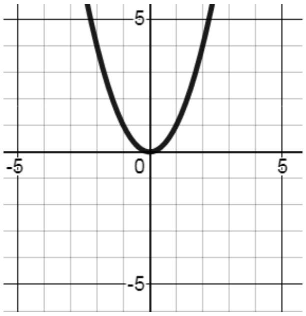
Later in your course, you will be asked what x-values are allowed in this formula. It may not be obvious, but x can take on <u>any</u> value; that is, x can be any real number.

As for y-values, the graph might show that y can't be any smaller than 2. That is, y has to be <u>at least</u> 2; we could even write $y \geq 2$.

The graph is certainly not a line; it's in the shape of a "V." It is sharp at its bottom point (3, 2), not smooth and curvy like the graph of the parabola $y = x^2$ that you may have seen before.



$y = |x|$



$y = x^2$

Homework

1. Graph: $y = |x|$

2. Graph: $y = |x| + 3$

3. Graph: $y = |x| - 4$

4. Graph: $y = |x + 2|$

5. Graph: $y = |x - 5|$

6. Graph: $y = |x + 3| - 4$

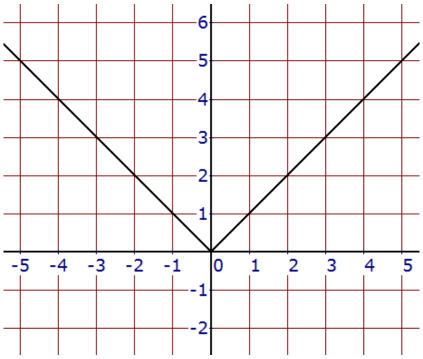
The absolute value of a positive number is itself.

The absolute value of 0 is 0.

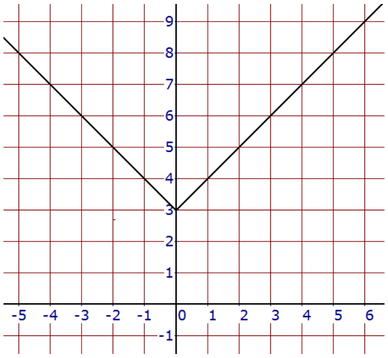
The absolute value of a negative number is its opposite.

Solutions

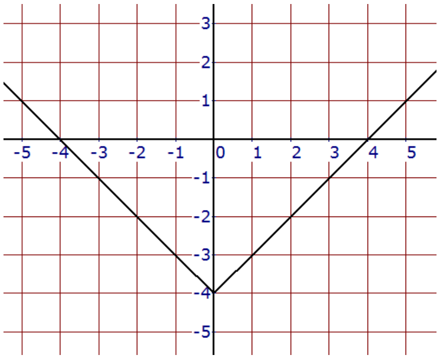
1.



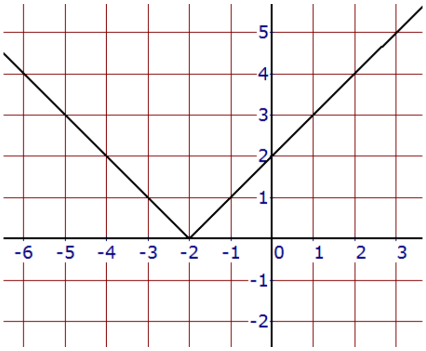
2.



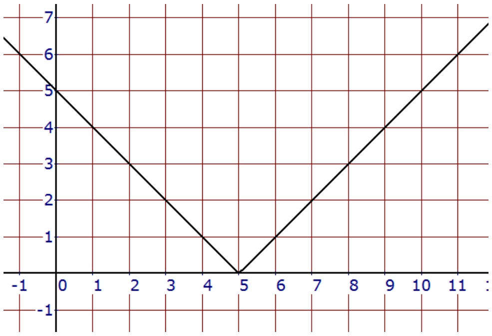
3.



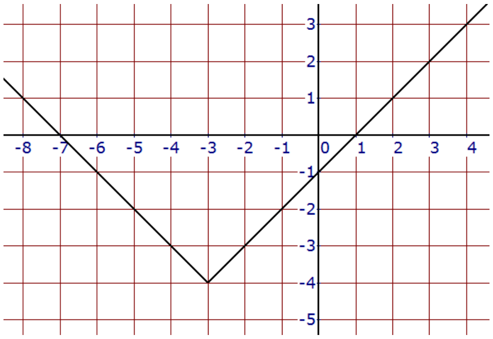
4.



5.



6.



"The aim of education is the knowledge, not of facts, but of values." – William S. Burroughs

CH NN – BREAK-EVEN POINT, LINEAR FUNCTIONS

Certainly the ultimate goal of a business is not merely to "break even"; however, the **break-even point** is one of the most important tools concepts in business. It tells the business owner the point (in production, time, or investments) where losses have ended and profits will appear (or, unfortunately, the other way around). Every business which requires funding (either from investments, like selling stock, or the borrowing of money) requires a written business plan stating the projected break-even time.

❑ *BUSINESS TERMS*

A few business terms, with simplified definitions, will help us understand how algebra can represent the real world. We'll use the term ***Revenue*** to represent all the money that a company takes in through its sales and services. The term ***Cost*** represents the money spent by the company to produce those sales and services. And we'll define ***Profit*** to be the difference between revenue and cost. We're ready to write a formula now:

Profit = Revenue – Cost

$$\boxed{P = R - C}$$

❑ REVIEW OF INEQUALITIES

The fact that 3 is **less than** 5 is written

$$3 < 5$$

If we want to say that x is **less than or equal to** 5, we write

$$x \leq 5$$

Similarly, we might write $10 > 8$ to say that 10 is **greater than** 8, and we write $n \geq 15$ to say that n is **greater than or equal to** 15. Notice that $7 \geq 7$ is a true statement because 7 is greater than or equal to 7. (Namely, it's equal to 7.)

Homework

1. a. Solve the profit formula $P = R - C$ for R.

   b. Solve the profit formula $P = R - C$ for C.

2. True/False:

   a. $7 < 9$ b. $-2 < 7$ c. $-3 < -7$ d. $-9 < -1$

   e. $8 \geq 8$ f. $0 > 9$ g. $0 > -9$ h. $-2 \leq -2$

   i. $-2 < -2$ j. $0 < 0$ k. $0 \geq 0$ l. $-12 > -5$

   m. $\pi > \sqrt{2}$ n. $2\pi \geq \pi$ o. $\sqrt{2} < \sqrt{3}$ p. $\sqrt{10} < \pi$

3. Consider the inequality $n \geq 7$. Which of the following values of n would make the statement true?

   a. 15 b. 7.001 c. 6.999 d. 7 e. 2 f. -3 g. -29 h. $\sqrt{40}$

4. Which of the following values of x will satisfy the inequality $x < -3$?

   a. -100 b. $-\pi$ c. -3 d. -2.999 e. 0 f. 3.14 g. 140

❑ DEFINITION OF BREAK-EVEN

There are two ways to define the ***break-even point***. One is to say that break-even occurs when revenue matches cost; that is, when $R = C$. On the other hand, if the revenue and cost are the same, then there's zero profit. (See that?) So break-even can also be defined as the point at which the profit is 0.

Either equation can be used to find the ***break-even point***:

$$\boldsymbol{R = C}$$

$$\boldsymbol{P = 0}$$

To prove that the profit must be zero when revenue = cost, we can use a little algebra. Assume that

$$\begin{array}{lll} & R = C & \text{(one criterion for break-even)} \\ \Rightarrow & R - C = 0 & \text{(subtract } C \text{ from each side of the equation)} \\ \Rightarrow & \boldsymbol{P = 0} & \text{(substitute, since } P = R - C\text{)} \end{array}$$

❑ USING A TABLE

Assuming w represents the number of widgets a company manufactures and sells, let's assume the formula for revenue to be:

$$R = 4w$$

and a cost formula to go with it:

$$C = 2w + 8$$

The Revenue formula might come from the fact that we sell our widgets for $4 apiece.

The Cost formula may be the result of the fact that each widget costs $2 to manufacture, together with a fixed cost (salaries, rent, utilities, etc.) of $8.

Let's create a table with revenue, cost, and profit for various quantities of widgets sold, according to the given formulas.

w	$R = 4w$	$C = 2w + 8$	$P = R - C$
Widgets	Revenue	Cost	Profit
0	\$0	\$8	−\$8
1	\$4	\$10	−\$6
2	\$8	\$12	−\$4
3	\$12	\$14	−\$2
4	**\$16**	**\$16**	**\$0**
5	\$20	\$18	\$2
6	\$24	\$20	\$4

Break-Even Point →

Now for the important observations:

- ✓ As the number of widgets increases, so do the revenue and the cost. Also, the profit increases; that is, the numbers −8, −6, −4, −2, 0, 2, and 4 are getting larger.

- ✓ If we produce and sell anywhere from 0 to 3 widgets, we incur a loss (a negative profit). That is, when $w \le 3$, $P < 0$.

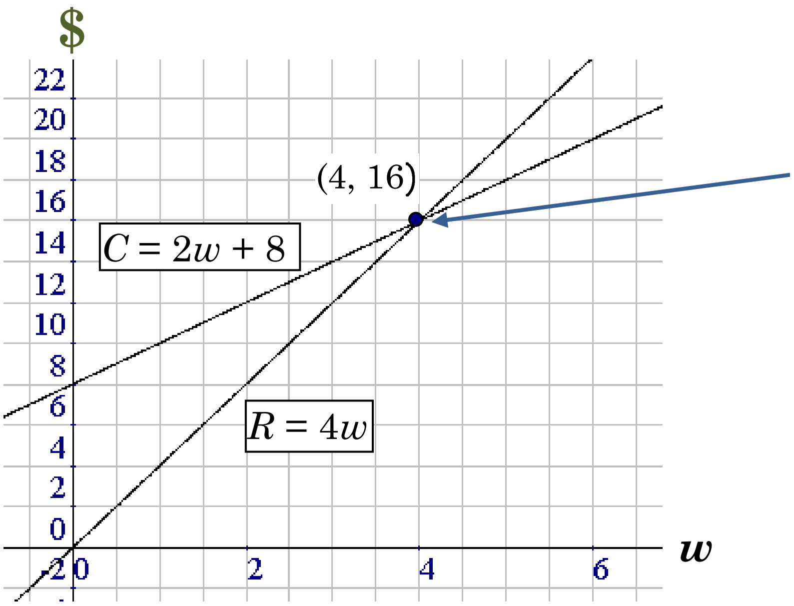
- ✓ At a production level of 4 widgets we break even. This is the point at which the revenue ($16) equals the cost ($16). But also note that this is where the profit is $0. Either way we look at it, according to the two definitions, <u>$w = 4$ is the break-even point</u>. When $w = 4$, $P = 0$.

- ✓ Any number of widgets beyond 4 (that is, 5 or more) produces a positive profit (we're making money!). Thus, if $w \geq 5$, then $P > 0$.

- ✓ In summary, a **loss** occurs when $w \leq 3$, the **break-even point** occurs when $w = 4$, and a **profit** results when $w \geq 5$.

❑ *USING A GRAPH*

We're going to make a picture of this situation of revenue, cost, profit, and the break-even point. Our grid will contain the graphs of both the revenue and cost formulas from the previous example. Note that the horizontal axis will certainly be w, the number of widgets, but the vertical axis will be the generic category money, since both revenue and cost are in units of money.



The ***break-even point*** occurs when *w* = 4, where both the revenue and the cost are $16.

Fewer than 4 widgets results in a loss — more than 4 produces a profit.

☐ USING AN EQUATION

The table and the graph were quite useful in determining and understanding the relationship among revenue, cost, profit, and break-even. But these take time to construct and they may give us only approximations. Let's use algebra to solve the same problem for the third time.

EXAMPLE 1: **The revenue formula is $R = 4w$ and the cost formula is $C = 2w + 8$. Find the break-even point.**

Solution: We recall that the break-even point occurs when the revenue equals the cost. Thus,

$R = C$ (to calculate break-even)

$4w = 2w + 8$ (substituting the given formulas)

$4w - \mathbf{2w} = 2w - \mathbf{2w} + 8$ (subtract $2w$ from each side)

$2w = 8$ (simplify each side)

$\frac{2w}{\mathbf{2}} = \frac{8}{\mathbf{2}}$ (divide each side by 2)

$w = 4$ (simplify)

Thus, the break-even point is $\boxed{\text{4 widgets}}$

just as we saw with the table and the graph.

EXAMPLE 2: **The revenue formula is $R = 12w - 20$ and the cost formula is $C = 2w + 30$.**

a) Calculate the profit formula.

b) Find the break-even point using the profit formula.

Solution: a) The profit formula is $P = R - C$, so we can calculate the profit like this:

$$P = R - C \quad \text{(the profit formula)}$$

$$\Rightarrow \quad P = (12w - 20) - (2w + 30) \quad \text{(notice the \underline{parentheses}!)}$$

$$\Rightarrow \quad P = 12w - 20 - 2w - 30 \quad \text{(distribute)}$$

$$\Rightarrow \quad \boxed{P = 10w - 50} \quad \text{(combine like terms)}$$

For part b) we find the break-even point by setting the profit formula to 0:

$$P = 0 \quad \text{(one criterion for break-even)}$$

$$\Rightarrow \quad 10w - 50 = 0 \quad \text{(substitute the given profit formula)}$$

$$\Rightarrow \quad 10w = 50 \quad \text{(add 50 to each side of the equation)}$$

$$\Rightarrow \quad \boxed{w = 5} \quad \text{(divide each side of the equation by 10)}$$

Homework

5. Regarding Example 2, part b), use the profit formula to show that we incur a loss if $w < 5$ and we enjoy a profit if $w > 5$. (A couple of examples will suffice.)

6. Suppose revenue and cost formulas are given by

$$R = 3w + 1 \qquad C = w + 5$$

a. Construct a table with columns for Widgets, Revenue, Cost, and Profit. Let w take the values from 0 to 4.

b. Use the table to determine the break-even point. Explain in <u>two</u> different ways how you arrived at your conclusion.

c. Graph both formulas on the same grid. Use the graphs to determine the break-even point.

d. Now find the break-even point by solving the formula equating Revenue and Cost.

7. Find the **break-even point** for the given revenue and cost formulas by solving an equation:

a. $R = 10w$ $\quad C = 7w + 18$

b. $R = 5w + 1$ $\quad C = 4w + 50$

c. $R = 8w - 9$ $\quad C = 3w + 6$

d. $R = 72w + 12$ $\quad C = 50w + 100$

8. Find the **profit** formula for the given revenue and cost formulas:

a. $R = 30w + 90$ $\quad C = 22w - 13$

b. $R = 22w - 5$ $\quad C = 10w + 17$

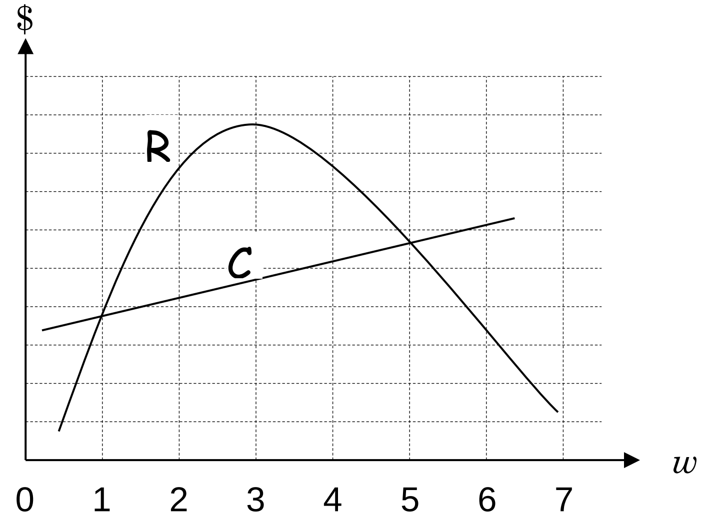
c. $R = w + 10$ $\quad C = 8w - 14$

d. $R = 13w$ $\quad C = 2w - 5$

e. $R = 99w + 17$ $\quad C = 9w$

f. $R = 8w - 1$ $\quad C = 7w - 10$

9. Consider the graphs of the revenue and cost formulas:

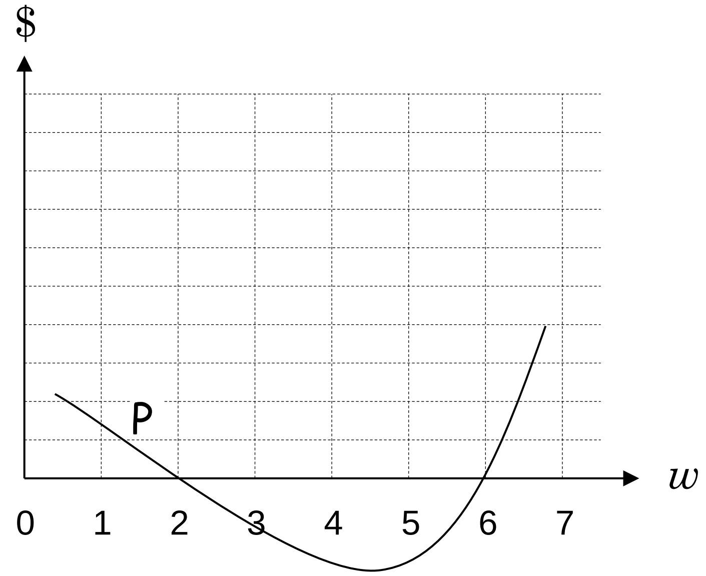


a. Find the two break-even points.

b. Is there a profit or loss when $w = 4$?

c. Is there a profit or loss when $w = 6$?

10. Consider the graph of the **profit** formula:



Find the two break-even points. What's happening between those two points?

11. Find the **break-even point** given each profit formula:

    a. $P = 18w - 810$
    b. $P = 2.5w - 300$
    c. $P = 5.23w - 287.65$
    d. $P = -7.6w + 410.4$

Review Problems

12. Let the revenue and cost formulas be given by $R = 7w + 1$ and $C = 5w + 11$. Construct a table with columns for widgets, revenue, cost, and profit, and let w take on the values from 3 to 6. Use the table to find the break-even point. Explain in two different ways how you arrived at your conclusion.

13. Use algebra (that is, solve an equation) to find the break-even point if the revenue and cost formulas are given by $R = 15w - 13$ and $C = 7w + 43$.

14. Find the profit formula if the revenue and cost formulas are given by $R = 10w + 13$ and $C = 6w - 5$.

15. Find the break-even point for the profit formula $P = 2.5w - 247.5$.

16. Graph $R = 3w - 4$ and $C = w + 1$ on the same grid, and then estimate the break-even point. How did you find that break-even point?

Solutions

1. a. $R = P + C$ b. $C = R - P$

2.
a. T b. T c. F d. T e. T f. F
g. T h. T i. F j. F k. T l. F
m. T n. T o. T p. F

3. a, b, d **4.** a, b

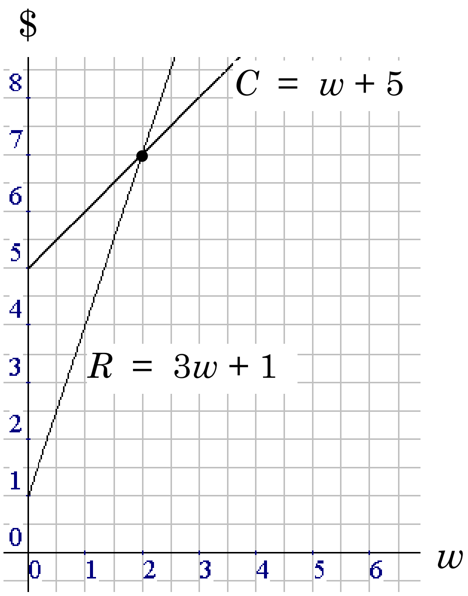
5. Suppose w is 3; then $P = 10(3) - 50 = 30 - 50 = -20$, a loss.
If w is 8, then $P = 10(8) - 50 = 80 - 50 = 30$, a profit.

6. a.

Widgets	Revenue	Cost	Profit
0	\$1	\$5	–\$4
1	\$4	\$6	–\$2
2	\$7	\$7	\$0
3	\$10	\$8	\$2
4	\$13	\$9	\$4

b. The break-even point is $w = 2$. It's the point where Revenue = Cost, and it's also the point where Profit = 0.

c.



d. The break-even point can be found by setting Revenue to Cost:

$R = C$	(to calculate break-even)
$3w + 1 = w + 5$	(substituting the given formulas)
$3w - \boldsymbol{w} + 1 = w - \boldsymbol{w} + 5$	(subtract w from each side)
$2w + 1 = 5$	(simplify)
$2w + 1 - \mathbf{1} = 5 - \mathbf{1}$	(subtract 1 from each side)
$2w = 4$	(simplify)
$\boldsymbol{w = 2}$	(divide each side by 2)

7. a. $w = 6$ b. $w = 49$ c. $w = 3$ d. $w = 4$

8. a. $P = R - C = (30w + 90) - (22w - 13) = 30w + 90 - 22w + 13$
$= 8w + 103$

b. $P = (22w - 5) - (10w + 17) = 22w - 5 - 10w - 17 = 12w - 22$

c. $P = -7w + 24$

d. $P = 11w + 5$

e. $P = 90w + 17$

f. $P = w + 9$

9. a. $w = 1$ and $w = 5$ (found by looking at the w-values where the graphs intersect)

b. profit, since the revenue graph is above the cost graph (i.e., the revenue exceeded the cost)

c. loss, since the cost graph is above the revenue graph (i.e., the cost exceeded the revenue)

10. $w = 2$ and $w = 6$ (found by looking at where the profit is zero; that is, where the profit graph crosses the w-axis, which is where $P = 0$)

The profit is negative (a loss) between the break-even points.

11. To find the break-even points, set each profit formula to zero:

a. $18w - 810 = 0 \Rightarrow 18w = 810 \Rightarrow w = 45$

b. $2.5w - 300 = 0 \Rightarrow 2.5w = 300 \Rightarrow w = 120$

c. $w = 55$

d. $w = 54$

12. $w = 5$. Why is this the break-even point? First, it's where $R = C$; second, it's where $P = 0$.

13. 7 widgets **14**. $P = 4w + 18$ **15**. 99 widgets

16. $w \approx 2.5$; it's the w-coordinate of the point of intersection

“Next in importance
to freedom and justice
is *education*, without
which neither
freedom nor justice
can be permanently
maintained.”

***James A. Garfield* (1831 - 1881)**
20th U.S. President

CH XX – BREAK-EVEN POINT, QUADRATIC

A while back we discussed the notions of revenue (*R*), cost (*C*), profit (*P*), and the break-even point:

Profit: $P = R - C$

Break-even: $R = C$ or $P = 0$

Now that we're getting proficient at factoring, we can solve more break-even business problems; these problems will result in a **quadratic equation**.

☐ BREAK-EVEN

EXAMPLE 1: **Find the break-even point(s) if the profit formula is given by $P = 2w^2 - 31w + 84$.**

Solution: We find the break-even points by setting the profit formula to zero:

$$2w^2 - 31w + 84 = 0 \quad \text{(set profit to 0)}$$

$$\Rightarrow \quad (2w - 7)(w - 12) = 0 \quad \text{(factor)}$$

$$\Rightarrow \quad 2w - 7 = 0 \text{ or } w - 12 = 0 \quad \text{(set each factor to 0)}$$

$$\Rightarrow \quad 2w = 7 \text{ or } w = 12 \quad \text{(solve each equation)}$$

$$\Rightarrow \quad w = 3\frac{1}{2} \text{ or } w = 12$$

Thus, the break-even points are

$$\boxed{3\frac{1}{2}\text{ widgets and 12 widgets}}$$

Of course, $3\frac{1}{2}$ widgets can't really exist, but it's good enough for this problem. But if w stood for wages, for instance, then $3\frac{1}{2}$ would make sense, since that number represents \$3.50, a perfectly legit answer.

EXAMPLE 2: **Find the break-even point(s) if revenue and cost are given by the formulas**

$$\boldsymbol{R = 3w^2 - 3w - 8}$$
$$\boldsymbol{C = 2w^2 + 30w - 268}$$

Solution: Recall that one of the two ways to describe the **break-even points** is by equating revenue and cost. Notice that we put the resulting equation in standard form by bringing all the terms on the right side to the left side so that the right side will be zero.

$R = C$ (to find break-even)

$\Rightarrow$ $3w^2 - 3w - 8 = 2w^2 + 30w - 268$ (use the given formulas)

$\Rightarrow$ $w^2 - 3w - 8 = 30w - 268$ (subtract $2w^2$)

$\Rightarrow$ $w^2 - 33w - 8 = -268$ (subtract $30w$)

$\Rightarrow$ $w^2 - 33w + 260 = 0$ (add 268 $\Rightarrow$ standard form)

$\Rightarrow$ $(w - 20)(w - 13) = 0$ (factor)

$\Rightarrow$ $w - 20 = 0$ or $w - 13 = 0$ (set each factor to 0)

$\Rightarrow$ $w = 20$ or $w = 13$ (solve each equation)

And so the two break-even points are

$$\boxed{\text{20 widgets and 13 widgets}}$$

Homework

1. Find the **break-even points** for the given profit formula:

   a. $P = w^2 - 12w + 35$ b. $P = 2w^2 - 13w + 15$

   c. $P = w^2 - 25w + 150$ d. $P = 6w^2 - 31w + 40$

2. Find the **break-even points** given the revenue and cost formulas:

   a. $R = 2w^2 + 8w - 20$ $C = w^2 + 24w - 75$

   b. $R = 2w^2 - 3w + 1$ $C = -w^2 + 16w - 29$

   c. $R = 5w^2 - 26w + 80$ $C = 3w^2 + w + 10$

Review Problems

3. If the profit formula is given by $P = w^2 - 57w + 350$, find the two break-even points.

4. If the revenue and cost are given by the formulas $R = 5w^2 - 30w + 100$ and $C = 4w^2 + 4w - 180$, find the break-even points.

Solutions

1. a. $w = 5, 7$ b. $w = \frac{3}{2}, 5$ c. $w = 10, 15$ d. $w = \frac{8}{3}, \frac{5}{2}$

2. a. $w = 5, 11$ b. $w = \frac{10}{3}, 3$ c. $w = \frac{7}{2}, 10$

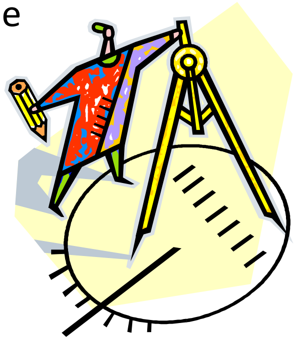
3. $w = 7$ and $w = 50$ **4**. $w = 14$ and $w = 20$

"There is one purpose to life and one only: to bear witness to and understand as much as possible of the complexity of the world – its beauty, its mysteries, its riddles. The more you understand, the more you look – the greater is your enjoyment of life and your sense of peace. That's all there is to it. If an activity is not grounded in '***to love***' or '***to learn,***' it does not have value."

– Anne Rice, American Author

CH NN –THE CIRCLE, CENTER *AT* THE ORIGIN

We're now ready for a new type of graph. In this chapter, we analyze "nature's perfect shape," the circle. Whereas the equation of a line has no squared variables, and a parabola has one squared variable, we will see that the equation of a circle has both variables squared.

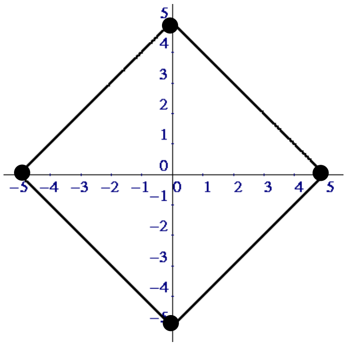


❑ *GRAPHING A CIRCLE*

EXAMPLE 1: **Graph:** $x^2 + y^2 = 25$

Solution: The graph is undoubtedly not a line, since the variables are squared. Let's plot points and see what we get.

Let's first check out the **intercepts**. If we set $x = 0$, the resulting equation is $y^2 = 25$, whose two solutions are $y = \pm 5$. Thus, there are two y-intercepts, **(0, 5)** and **(0, –5)**. You can set y to 0 and calculate the x-intercepts to be **(5, 0)** and **(–5, 0)**. We now have four points on our graph, but it's unclear how to connect them — maybe the graph looks like a diamond? We'll find some other points; for example, if we let $x = 3$, then



Are four points enough to make an accurate graph? Perhaps, perhaps not.

$$3^2 + y^2 = 25 \quad \Rightarrow \quad 9 + y^2 = 25$$

$$\Rightarrow \quad y^2 = 16 \quad \Rightarrow \quad y = \pm 4$$

$\Rightarrow$ **(3, 4)** and **(3, –4)** are on the graph.

If x is chosen to be -3, then

$$(-3)^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

⇒ **(–3, 4)** and **(–3, –4)** are also on the graph.

Now let $x = 4$:

$$4^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

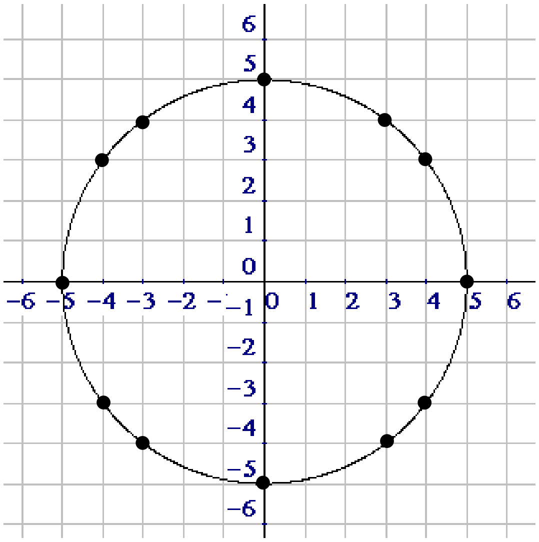
⇒ **(4, 3)** and **(4, –3)** are on the graph.

Our last choice for x will be -4:

$$(-4)^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

⇒ **(–4, 3)** and **(–4, –3)** are also on the graph.

This set of 12 points should be enough data to get a decent picture, which looks a lot like a circle:



Summary: The graph of

$$x^2 + y^2 = 25$$

is a circle with its center at the origin, (0, 0), and with a radius of 5.

Homework

1. For each circle, determine the four **intercepts**:

a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 2$ c. $x^2 + y^2 = 4$

d. $x^2 + y^2 = 7$ e. $x^2 + y^2 = 49$ f. $x^2 + y^2 = 60$

2. Given the circle and the *x*-value (i.e., an input), find the *y*-values (i.e., the outputs):

a. $x^2 + y^2 = 100;\ x = 6$ b. $x^2 + y^2 = 100;\ x = -8$

c. $x^2 + y^2 = 1;\ x = 1$ d. $x^2 + y^2 = 169;\ x = 5$

e. $x^2 + y^2 = 169;\ x = -12$ f. $x^2 + y^2 = 169;\ x = -5$

g. $x^2 + y^2 = 10;\ x = 2$ h. $x^2 + y^2 = 12;\ x = -2$

3. Consider again the circle $x^2 + y^2 = 25$ from Example 1. Its center is the origin and its radius is 5. Now look at the graph and notice that there's no point on the graph with an x-value of 8. Prove this fact algebraically (that is, use the equation).

EXAMPLE 2:

A. Consider the circle $x^2 + y^2 = 81$. Using Example 1 as a guide, we infer that its center is the **origin** and its radius is **9**.

B. Now look at the circle $x^2 + y^2 = 13$. The center is **(0, 0)**, and the radius is $\boldsymbol{\sqrt{13}}$.

C. If the center of a circle is the origin, and if its radius is 15, what is the equation of the circle? It's $\boldsymbol{x^2 + y^2 = 225}$.

D. What is the equation of the circle with center (0, 0) and radius $3\sqrt{7}$?

$$x^2 + y^2 = \left(3\sqrt{7}\right)^2, \text{ which is } \boldsymbol{x^2 + y^2 = 63}.$$

Note: $\left(3\sqrt{7}\right)^2 = 3^2 \cdot \sqrt{7}^2 = 9 \cdot 7 = \mathbf{63}$

Homework

4. Find the center and radius of each circle:

a. $x^2 + y^2 = 25$ b. $x^2 + y^2 = 144$

c. $x^2 + y^2 = 1$ d. $x^2 + y^2 = 17$

e. $x^2 + y^2 = 27$ f. $x^2 + y^2 = 200$

g. $x^2 + y^2 = 0$ h. $x^2 + y^2 = -9$

i. $x + y = 10$ j. $x^2 + y = 49$

5. Find the equation of the circle with center at the origin and the given radius:

a. $r = 10$ b. $r = 25$ c. $r = 1$

d. $r = 16$ e. $r = \sqrt{11}$ f. $r = \sqrt{18}$

g. $r = 4\sqrt{5}$ h. $r = 3\sqrt{7}$

Note: Working with the radius can be confusing. If given the circle equation in standard form, the radius of the circle is found by taking the positive square root of the number to the right of the equal sign. On the other hand, if you know the radius, you square it when you put it into the formula.

6. The ***unit circle*** is the circle whose *center* is at the origin and whose *radius* is 1.

a. What are the coordinates of the center of the unit circle?

b. What is the radius of the unit circle?

c. What is the equation of the unit circle?

d. What is the area of the unit circle?

e. What is the circumference of the unit circle?

7. Consider the circle $x^2 + y^2 = 7$.

   a. What is the radius?
   b. What is the diameter?
   c. What is the circumference?
   d. What is the area?

8. Sketch a circle (whose center is not necessarily at the origin) that has exactly

   a. 4 intercepts
   b. 2 intercepts
   c. 3 intercepts
   d. 1 intercept
   e. 0 intercepts

9. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation for each situation:

   a. $k > 0$ b. $k = 0$ c. $k < 0$

Review Problems

10. Find the center and radius of the circle $x^2 + y^2 = 20$.

11. True/False:

   a. Every circle has at least one intercept.
   b. The radius of the circle $x^2 + y^2 = 1$ is 1.
   d. $x^2 + y^2 + 4 = 3$ is a circle.
   e. $x^2 + y^2 = 1$ is called the *unit circle*.
   f. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $(\pm 1, 0)$ $(0, \pm 1)$ b. $(\pm\sqrt{2}, 0)$ $(0, \pm\sqrt{2})$

c. $(\pm 2, 0)$ $(0, \pm 2)$ d. $(\pm\sqrt{7}, 0)$ $(0, \pm\sqrt{7})$

e. $(\pm 7, 0)$ $(0, \pm 7)$ f. $(\pm 2\sqrt{15}, 0)$ $(0, \pm 2\sqrt{15})$

2. a. ± 8 b. ± 6 c. 0 d. ± 12

e. ± 5 f. ± 12 g. $\pm\sqrt{6}$ h. $\pm 2\sqrt{2}$

3. Hint: Let $x = 8$ in the circle equation and try to solve for y.

4. a. C(0, 0) $r = 5$ b. C(0, 0) $r = 12$ c. C(0, 0) $r = 1$

d. C(0, 0) $r = \sqrt{17}$ e. C(0, 0) $r = 3\sqrt{3}$ f. C(0, 0) $r = 10\sqrt{2}$

g. It's not a circle; the graph is just the origin.

h. It's not a circle; there are two reasons. First, the radius would be $\sqrt{-9}$, which is not a real number. Second, if two numbers are squared and then added together, there's no way that sum could be negative.

i. It's a line, not a circle.

j. It's not a circle — do you know what it is?

5. a. $x^2 + y^2 = 100$ b. $x^2 + y^2 = 625$ c. $x^2 + y^2 = 1$

d. $x^2 + y^2 = 256$ e. $x^2 + y^2 = 11$ f. $x^2 + y^2 = 18$

g. $x^2 + y^2 = 80$ h. $x^2 + y^2 = 63$

6. a. (0, 0) b. 1 c. $x^2 + y^2 = 1$

d. $A = \pi r^2 = \pi(1^2) = \pi(1) = \boldsymbol{\pi}$ e. $C = 2\pi r = 2\pi(1) = \boldsymbol{2\pi}$

7. a. $r = \sqrt{7}$ b. $d = 2\sqrt{7}$ c. $C = 2\sqrt{7}\,\pi$ d. $A = 7\pi$

8. You're on your own.

9. a. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.

b. If $k = 0$, the graph is just the single point (0, 0); i.e., the origin.

c. If $k < 0$, the graph is empty (there's no graph at all).

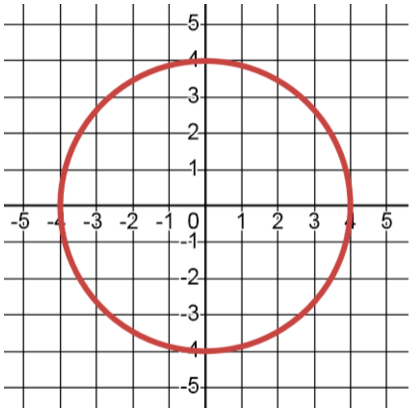
10. C(0, 0); $r = 2\sqrt{5}$

11. a. F b. T d. F e. T f. T

“Only those who dare
to fail greatly
can ever achieve greatly.”

– Robert F. Kennedy

CH XX – THE CIRCLE, CENTER *OFF* THE ORIGIN



You've graphed circles whose centers are at the origin in the Chapter *The Circle – Center at the Origin*. For example, the center of the circle $x^2 + y^2 = 16$ is (0, 0), and its radius is 4. Our goal in this chapter is to allow the center of a circle to be somewhere else in the plane.

What kind of equation would result in such a circle? The example in the next section,

$$(x - 2)^2 + (y + 1)^2 = 25$$

will provide us with one. But before we get there, we need to understand why this equation is likely a circle: If you square (expand) the binomials, note that you'll get an x^2 term, a y^2 term, some x's, some y's, and some numbers. The squared terms lead us to believe that the graph of the equation is a circle, but that's only a theory.

☐ *USING THE ANNIHILATOR METHOD*

<u>EXAMPLE 1:</u> **Graph:** $(x - 2)^2 + (y + 1)^2 = 25$

<u>Solution:</u> First, it might be clear (due to the –2 and the 1) that the center of the circle is <u>not</u> the origin. Second, our best guess right now is that the radius is 5 (since 5 is the positive square root of 25). Let's check out these ideas.

It's hard to know what values of x we should choose to find our points to plot, but here's a neat trick to find four useful points.

We'll start with <u>$x = 2$</u> (this rids us of the first term), and then solve for $\boldsymbol{y}$ in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

$$\begin{aligned} x = \mathbf{2} \quad &\Rightarrow \quad (\mathbf{2} - 2)^2 + (y + 1)^2 = 25 \\ &\Rightarrow \quad 0^2 + (y + 1)^2 = 25 \\ &\Rightarrow \quad (y + 1)^2 = 25 \\ &\Rightarrow \quad y + 1 = \pm 5 \qquad \text{[Don't forget: 25 has \underline{2} square roots]} \\ &\Rightarrow \quad y = -1 \pm 5 \\ &\Rightarrow \quad y = 4 \text{ or } -6 \end{aligned}$$

Since letting $x = 2$ produced two y-values, we have the two points **(2, 4)** and **(2, –6)** on our circle.

Next, we'll let <u>$y = -1$</u> (this annihilates the second term), and then solve for $\boldsymbol{x}$ in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

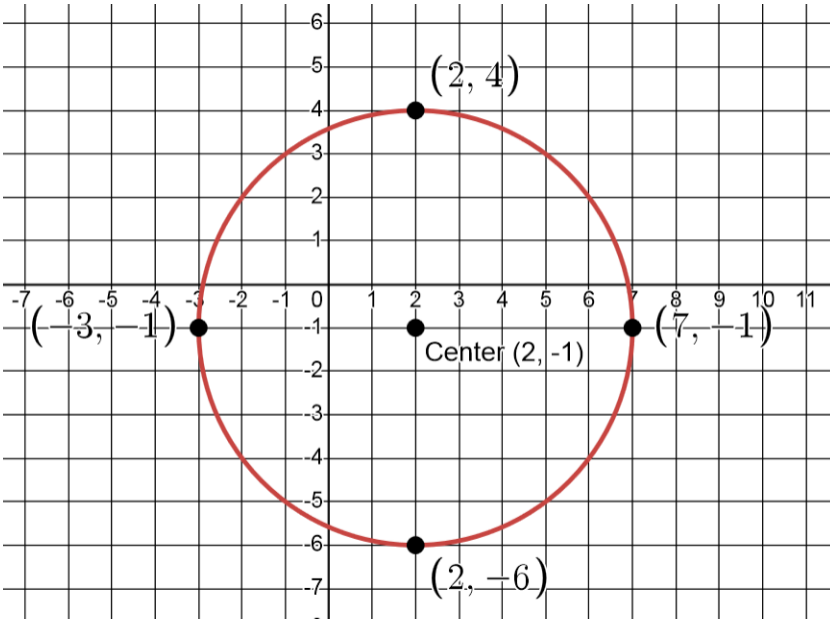
$$\begin{aligned} y = \mathbf{-1} \quad &\Rightarrow \quad (x - 2)^2 + (\mathbf{-1} + 1)^2 = 25 \\ &\Rightarrow \quad (x - 2)^2 + 0^2 = 25 \\ &\Rightarrow \quad (x - 2)^2 = 25 \\ &\Rightarrow \quad x - 2 = \pm 5 \\ &\Rightarrow \quad x = 2 \pm 5 \\ &\Rightarrow \quad x = 7 \text{ or } -3 \end{aligned}$$

By letting $y = -1$, we determined that two more points on the circle are **(7, –1)** and **(–3, –1)**.

What have we done here? By cleverly choosing values of *x* and *y* (do you see how we chose them?), we have discovered these four points on our circle:

(2, 4) (2, –6) (7, –1) (–3, –1)

Let's plot these four points, connect them to make a circle, and then figure out the circle's center and radius.



Can you see that the center of the circle is the point (2, –1) and that the radius is 5? Let's summarize:

The circle $(x-2)^2 + (y+1)^2 = 25$ has its center at (2, –1) and has a radius of 5.

Homework

1. Find the **center** and **radius** of each circle by mimicking Example 1 (the *annihilator method*):

a. $x^2 + y^2 = 64$

b. $x^2 + y^2 = 24$

c. $x^2 + (y - 7)^2 = 4$

d. $(x - 3)^2 + y^2 = 5$

e. $(x + 1)^2 + (y + 8)^2 = 60$

f. $(x - 2)^2 + (y - 3)^2 = 99$

☐ *A QUICKER APPROACH*

Look at the problem and the solution in Example 1. Do you see any connections there? The x-coordinate of the center (the 2) is the *opposite* of the number following the x in the circle equation. Also, the y-coordinate of the center (the -1) is the *opposite* of the number following the y in the circle equation. And last, the circle's radius, 5, is the positive square root of the 25 on the right side of the circle equation (just as we expected).

The following table shows the relationships between the equations of some circles and the circles' centers and radii. Study it carefully.

Equation of Circle	Center	Radius
$(x - 7)^2 + (y - 4)^2 = 121$	(7, 4)	11
$(x + 3)^2 + (y + 5)^2 = 49$	(−3, −5)	7
$(x - 1)^2 + (y + 12)^2 = 21$	(1, −12)	$\sqrt{21}$
$(x + 11)^2 + (y - 9)^2 = 93$	(−11, 9)	$\sqrt{93}$

Equation of Circle	Center	Radius
$x^2 + (y - 8)^2 = 1$	(0, 8)	1
$(x + 13)^2 + y^2 = 50$	(–13, 0)	$5\sqrt{2}$
$x^2 + y^2 = 108$	(0, 0)	$6\sqrt{3}$

Homework

2. Find the **center** and **radius** of each circle using the "shortcut" described in the chart above:

a. $(x + 5)^2 + (y - 3)^2 = 144$

b. $(x - 1)^2 + (y + 11)^2 = 48$

c. $(x + 7)^2 + (y + \pi)^2 = 50$

d. $(x - \sqrt{2})^2 + (y - 2.3)^2 = 1$

e. $(x + 3) + (y - 1) = 9$

f. $(x - 3)^2 + (y + 2) = 7$

3. Find the **equation** of the circle with the given *center* and *radius*:

a. C(0, 0) $r = 7$

b. C(0, 0) $r = \sqrt{10}$

c. C(3, 0) $r = 3$

d. C(0, 4) $r = 1$

e. C(0, –2) $r = \sqrt{3}$

f. C(–12, 0) $r = 12$

g. C(2, 7) $r = 10$

h. C(–1, –3) $r = 2\sqrt{3}$

i. C(3, –4) $r = 3\sqrt{5}$

j. C(–2, 9) $r = 5\sqrt{7}$

k. C(1, 2) $r = 0$

l. C(–3, 5) $r = -9$

❑ *A NEW TWIST FOR THE CIRCLE*

Now for a tricky circle question:

Find the center and radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.

This circle is not in the standard form we've been using to extract the center and radius. So, how in the heck do we convert this circle equation containing no parentheses into the proper form with the two sets of parentheses? Any ideas before you read on?

EXAMPLE 2: **Graph the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.**

Solution: Creating the graph will depend on finding the circle's center and radius. To calculate these, we need to convert the given equation of the circle into standard form.

Remember the "magic number" we used to solve quadratic equations by completing the square? We use the same trick here, except we will complete the square in both variables. Cool . . . we get to calculate two magic numbers!

Start with the given equation of the circle:

$$x^2 + y^2 + 8x - 6y + 9 = 0$$

Rearrange the terms, putting the x-terms next to each other and the y-terms next to each other:

$$x^2 + 8x + y^2 - 6y + 9 = 0$$

Take the constant 9 to the other side of the equation:

$$x^2 + 8x + y^2 - 6y = -9$$

Now calculate the two "magic numbers":

Half of 8 is 4, and $4^2 = \mathbf{16}$. This is the magic number for x.

Half of –6 is –3, and $(-3)^2 = \mathbf{9}$. This is the magic number for y.

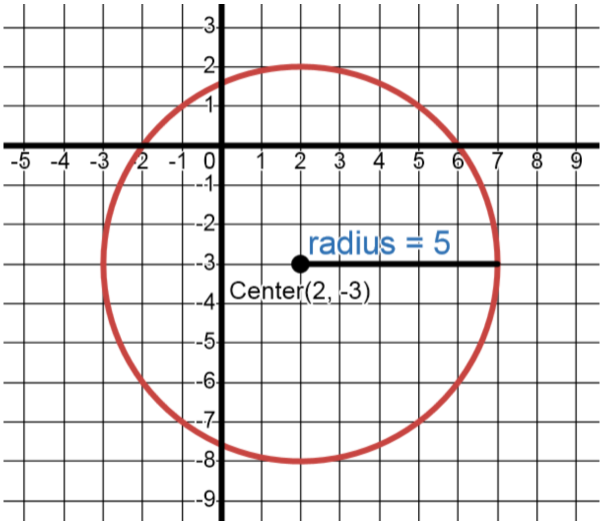
Now we add the magic numbers to both sides of the equation so that perfect square trinomials will be formed on the left side:

$$\underbrace{x^2+8x+\boxed{16}}_{\text{factorable}}+\underbrace{y^2-6y+\boxed{9}}_{\text{factorable}} = -9+\boxed{16}+\boxed{9}$$

Now factor the first three terms, then factor the next three terms, and then do the arithmetic on the right side of the equation:

$$(x+4)^2+(y-3)^2 = 16$$

We made it! Now that the circle equation is in standard form, we can read the center and radius directly from the equation. The center is (–4, 3) and the radius is 4 (the positive square root of 16). This info is just what we need to graph our circle:



Homework

4. Find the **center** and **radius** of each circle:

   a. $x^2+y^2+8x-10y-40 = 0$

   b. $x^2+y^2-16x+8y+31 = 0$

   c. $x^2+y^2+12x+14y+49 = 0$

d. $x^2 + y^2 + 14x + 24 = 0$

e. $x^2 + y^2 - 6y - 216 = 0$

f. $x^2 + y^2 + 8x + 6y + 10 = 0$

g. $x^2 + y^2 - 2x - 4y - 3 = 0$

h. $x^2 + y^2 + 4x - 10y + 12 = 0$

i. $x^2 + y^2 - 20x + 6y + 101 = 0$

j. $x^2 + y^2 + 2x - 4y + 25 = 0$

Note: If you come across a circle equation like

$$3x^2 + 3y^2 + 24x - 15y + 6 = 0$$

just divide each side of the equation by 3 (meaning divide every term by 3), and then proceed as usual.

Review Problems

5. Describe the graph of each equation:

   a. $x^2 + y^2 = 1{,}000{,}000$ b. $x^2 + y^2 = 1$

   c. $x^2 + y^2 = 0$ d. $x^2 + y^2 = -9$

6. Consider the equation $(x - h)^2 + (y - k)^2 = C$. Describe the graph of this equation

   a. if $C > 0$ b. if $C = 0$ c. if $C < 0$

7. Find the center and radius of the circle $x^2 + y^2 = 20$.

8. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

9. Matching:

   ____ $y = 3$ A. parabola

   ____ $2x^2 - 3y = 10$ B. circle

   ____ $x^2 + (y - 1)^2 = 2$ C. horizontal line

   ____ $x - y + 10 = 0$ D. non-horizontal line

10. True/False:

a. Every circle has at least one intercept.

b. The radius of the circle $x^2 + y^2 = 1$ is 1.

c. The graph of $y = x^2 - 9x + 17$ is a parabola.

d. $x^2 + y^2 + 4 = 3$ is a circle.

e. $x^2 + y^2 = 1$ is called the *unit circle*.

f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.

g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.

h. The graph of $10x^2 + 10y^2 = 37$ is a circle.

i. The graph of $10x^2 - 10y^2 = 37$ is a circle.

j. The graph of $10x^2 + 9y^2 = 37$ is a circle.

k. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. C(0, 0) $r = 8$
b. C(0, 0) $r = 2\sqrt{6}$
c. C(0, 7) $r = 2$
d. C(3, 0) $r = \sqrt{5}$
e. C(–1, –8) $r = 2\sqrt{15}$
f. C(2, 3) $r = 3\sqrt{11}$

2. a. C(–5, 3) $r = 12$
b. C(1, –11) $r = 4\sqrt{3}$
c. C(–7, –π) $r = 5\sqrt{2}$
d. C($\sqrt{2}$, 2.3) $r = 1$
e. Trick: It's a line
f. Trick: It's a parabola

3. a. $x^2 + y^2 = 49$
b. $x^2 + y^2 = 10$
c. $(x - 3)^2 + y^2 = 9$
d. $x^2 + (y - 4)^2 = 1$
e. $x^2 + (y + 2)^2 = 3$
f. $(x + 12)^2 + y^2 = 144$
g. $(x - 2)^2 + (y - 7)^2 = 100$
h. $(x + 1)^2 + (y + 3)^2 = 12$

i. $(x - 3)^2 + (y + 4)^2 = 45$

j. $(x + 2)^2 + (y - 9)^2 = 175$

k. Not a circle; just the point (1, 2)

l. Not a circle; in fact, no graph at all

4. a. C(−4, 5) $r = 9$

b. C(8, −4) $r = 7$

c. C(−6, −7) $r = 6$

d. C(−7, 0) $r = 5$

e. C(0, 3) $r = 15$

f. C(−4, −3) $r = \sqrt{15}$

g. C(1, 2) $r = 2\sqrt{2}$

h. C(−2, 5) $r = \sqrt{17}$

i. C(10, −3) $r = 2\sqrt{2}$

j. Not a circle

5. a. A circle with center at the origin and a radius of 1,000.

b. A circle with center at the origin and a radius of 1.

c. The point (0, 0), and that's it.

d. Since the sum of two squares is <u>never</u> negative, there is NO graph.

6. a. If $C > 0$, the graph is a circle with center at the origin and radius $\sqrt{C}$.

b. If $C = 0$, the graph is just the single point (h, k); i.e., the "center" of a phantom circle.

c. If $C < 0$, the graph is empty (there's no graph at all).

7. C(0, 0); $r = 2\sqrt{5}$

8. C(5, −1); $r = \sqrt{23}$

9. C, A, B, D

10. a. F b. T c. T d. F e. T f. F
g. T h. T i. F j. F k. T

"The educated differ from the uneducated as much as the living from the dead."

Αριστοτλε (Aristotle)

CH XX –PREPARING FOR COMPLETING THE SQUARE

☐ *INTRODUCTION*

I'm assuming that the only methods you might have learned for solving a *quadratic equation* are the **factoring method (**and perhaps the **Quadratic Formula)**. For example, to solve the quadratic equation

$$2x^2 + 3x - 20 = 0$$

by factoring, you would proceed as follows:

$$2x^2 + 3x - 20 = 0 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad (2x - 5)(x + 4) = 0 \qquad \text{(factor the quadratic)}$$

$$\Rightarrow \quad 2x - 5 = 0 \quad \text{OR} \quad x + 4 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad x = \frac{5}{2} \quad \text{OR} \quad x = -4\text{, and we're done — two solutions.}$$

So factoring is a good enough method, you say? I cannot agree with you; I submit to you the following quadratic equation:

$$x^2 + 7x + 5 = 0$$

It's not a complicated equation — the numbers are small, and there aren't even any negative numbers to mess with. So go ahead; I dare you to factor that thing.

See my point? Trust me, that equation <u>does</u> have two solutions, but factoring is not the method that will yield those two solutions. We need a new method for solving quadratic equations, and this chapter will give you some of the prerequisite skills required for a future chapter that covers the technique called *Completing the Square*.

☐ FACTORING PERFECT SQUARE TRINOMIALS

Let's review the term ***perfect square***. The number 100 is a perfect square because 100 can be written as the square of 10: $100 = 10^{\mathbf{2}}$. Also, the expression $(x + 5)^{\mathbf{2}}$ is a perfect square, since it is the square of $x + 5$. Even n^6 is a perfect square because it is the square of n^3: $n^6 = (n^3)^{\mathbf{2}}$.

One of the steps in solving a quadratic equation by *completing the square* is factoring a ***perfect square trinomial***. Let's look at four examples.

Example 1:

$x^2 + 14x + 49$ is a perfect square trinomial because it's a trinomial that factors into the square of a binomial:

$$x^2 + 14x + 49 \;=\; (x + 7)(x + 7) \;=\; (x + 7)^2$$

[perfect square trinomial] [square of a binomial]

Example 2:

$n^2 - 20n + 100$ is a perfect square trinomial:

$$n^2 - 20n + 100 \;=\; (n - 10)(n - 10) \;=\; (n - 10)^2$$

Here are a couple of examples with fractions:

Example 3:

$$a^2 + 3a + \frac{9}{4} \;=\; \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) \;=\; \left(a + \frac{3}{2}\right)^2$$

Check:

$$\left(a + \frac{3}{2}\right)^2 \;=\; \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) \;=\; a^2 + \underbrace{\frac{3}{2}a + \frac{3}{2}a}_{\frac{3}{2}+\frac{3}{2}=\frac{6}{2}=3} + \frac{9}{4} \;=\; a^2 + 3a + \frac{9}{4} \checkmark$$

Example 4:

$$y^2 - \frac{2}{5}y + \frac{1}{25} = \left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = \left(y - \frac{1}{5}\right)^2$$

Check:

$$\left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = y^2 - \frac{1}{5}y - \frac{1}{5}y + \frac{1}{25} = y^2 - \frac{2}{5}y + \frac{1}{25} \checkmark$$

Homework

1. Factor each perfect square trinomial:

a. $x^2 + 10x + 25$

b. $y^2 - 18y + 81$

c. $a^2 + a + \frac{1}{4}$

d. $m^2 - \frac{4}{3}m + \frac{4}{9}$

e. $z^2 + \frac{2}{5}z + \frac{1}{25}$

f. $w^2 - \frac{5}{3}w + \frac{25}{36}$

g. $b^2 + \frac{9}{5}b + \frac{81}{100}$

h. $u^2 + \frac{3}{2}u + \frac{9}{16}$

i. $n^2 - \frac{4}{7}n + \frac{4}{49}$

j. $x^2 + \frac{10}{11}x + \frac{25}{121}$

☐ *REVIEW OF SOLVING QUADRATICS BY TAKING SQUARE ROOTS*

Another skill required for completing the square is taking square roots to solve quadratic equations. You might want to review *Quadratic Equations by Taking Square Roots*. Here's an example from that chapter.

EXAMPLE 5: **Solve the quadratic equation: $(x+7)^2 = 81$**

Solution: According to the Square Root Theorem, we can remove the squaring by taking the square root of both sides of the equation, remembering that the number 81 has two square roots:

$$(x+7)^2 = 81 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad x+7 = \pm\sqrt{81} \qquad \text{(the Square Root Theorem)}$$

$$\Rightarrow \quad x+7 = \pm 9 \qquad (\sqrt{81} = 9)$$

$$\Rightarrow \quad x = -7 \pm 9 \qquad \text{(subtract 7 from each side)}$$

Remember!

81 has two square roots.

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$\boxed{x = 2 \text{ or } -16}$$

❑ THE "MAGIC NUMBER"

Consider the trinomial $x^2 + 10x + 25$. We've learned that its factorization is

$$x^2 + 10x + 25 = (x+5)^2\text{, which is the square of a binomial.}$$

Let's look carefully at the numbers in this equality. Notice that the 5 is half of the 10, and that the 25 is the square of the 5.

Let's do one more example. Consider the factorization

$$n^2 - 14n + 49 = (n-7)^2\text{, which is the square of a binomial.}$$

We note that the -7 is half of the -14, and that the 49 is the square of the -7.

So now imagine that I give you

$$x^2 + 6x$$

and I ask you to add a third term to this binomial so that the resulting trinomial will factor into the square of a binomial:

$$x^2 + 6x + \underline{???} \;=\; (x + ?)(x + ?) \;=\; (x + ?)^2$$

Each single "?" must be 3, since 3 is half of 6. Also, the "???" must be 9, since 9 is the square of the 3. In other words,

$$x^2 + 6x + \boxed{9} \;=\; (x + \underline{\mathbf{3}})(x + \underline{\mathbf{3}}) \;=\; (x + \underline{\mathbf{3}})^2$$

We shall call 9 the "magic number." It can be calculated for this problem using the following two-step rule:

1) Calculate half of 6, which is 3.
2) Square the 3, which is **9**, the "magic number."

Note: When we convert $x^2 + 6x$ to $x^2 + 6x + 9$ by adding the "magic number" **9**, we are not saying that they're equal, but there will always be a way, depending on the type of problem, of adding the magic number without violating any of the laws of algebra.

To become proficient in completing the square, we must be really good at finding the *magic number*. The following chart gives more examples of how this is done. We'll let *b* represent the number in front of the variable (the coefficient of the linear term).

Original Quadratic	Value of *b*	Half of *b*	Half of *b* Squared = **The Magic Number**	New Quadratic	New Quadratic in Factored Form
$x^2 + 22x$	22	11	**121**	$x^2 + 22x + \mathbf{121}$	$(x + 11)^2$
$A^2 - 14A$	–14	–7	**49**	$A^2 - 14A + \mathbf{49}$	$(A - 7)^2$
$n^2 + 9n$	9	$\frac{9}{2}$	$\mathbf{\frac{81}{4}}$	$x^2 + 9x + \mathbf{\frac{81}{4}}$	$\left(x + \frac{9}{2}\right)^2$
$y^2 - \frac{2}{5}y$	$-\frac{2}{5}$	$-\frac{1}{5}$	$\mathbf{\frac{1}{25}}$	$y^2 - \frac{2}{5}y + \mathbf{\frac{1}{25}}$	$\left(y - \frac{1}{5}\right)^2$
$3u^2 - 7u$	This problem can't be done yet; the leading coefficient is not 1.				

Homework

2. Find the **magic number** for each quadratic binomial:

a. $a^2 + 8a$
b. $x^2 - 12x$
c. $y^2 + 28y$
d. $b^2 + 3b$
e. $c^2 - 7c$
f. $t^2 + 11t$
g. $x^2 + x$
h. $y^2 - y$
i. $z^2 + 2z$
j. $g^2 + \frac{2}{3}g$
k. $a^2 - \frac{5}{7}a$
l. $x^2 - \frac{7}{13}x$

Review Problems

3. How many solutions does each quadratic equation have?

a. $(x + 3)(x + 4) = 0$ ___
b. $(x + 9)^2 = 0$ ___
c. $(x - 1)^2 = 10$ ___
d. $(x + 6)^2 = -9$ ___

4. Factor each trinomial:

a. $n^2 + 10n + 25$
b. $x^2 - 18x + 81$
c. $a^2 + 5a + \frac{25}{4}$
d. $x^2 - \frac{4}{3}x + \frac{4}{9}$
e. $y^2 + \frac{6}{7}y + \frac{9}{49}$
f. $t^2 - \frac{1}{5}t + \frac{1}{100}$

5. Find the magic number for $w^2 - \frac{7}{9}w$.

Solutions

1. a. $(x+5)^2$ b. $(y-9)^2$ c. $\left(a+\frac{1}{2}\right)^2$ d. $\left(m-\frac{2}{3}\right)^2$

e. $\left(z+\frac{1}{5}\right)^2$ f. $\left(w-\frac{5}{6}\right)^2$ g. $\left(b+\frac{9}{10}\right)^2$ h. $\left(u+\frac{3}{4}\right)^2$

i. $\left(n-\frac{2}{7}\right)^2$ j. $\left(x+\frac{5}{11}\right)^2$

2. a. $\frac{1}{2}\cdot 8 = 4$, and $4^2 = \mathbf{16}$ b. $\frac{1}{2}\cdot -12 = -6$, and $(-6)^2 = \mathbf{36}$

c. $\frac{1}{2}\cdot 28 = 14$, and $14^2 = \mathbf{196}$ d. $\frac{1}{2}\cdot 3 = \frac{3}{2}$, and $\left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$

e. $\frac{1}{2}\cdot -7 = -\frac{7}{2}$, and $\left(-\frac{7}{2}\right)^2 = \mathbf{\frac{49}{4}}$ f. $\frac{1}{2}\cdot 11 = \frac{11}{2}$, and $\left(\frac{11}{2}\right)^2 = \mathbf{\frac{121}{4}}$

g. $\frac{1}{2}\cdot 1 = \frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$ h. $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, and $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

i. $\frac{1}{2}\cdot 2 = 1$, and $1^2 = \mathbf{1}$ j. $\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$, and $\left(\frac{1}{3}\right)^2 = \mathbf{\frac{1}{9}}$

k. $\frac{1}{2}\cdot -\frac{5}{7} = -\frac{5}{14}$, and $\left(-\frac{5}{14}\right)^2 = \mathbf{\frac{25}{196}}$

l. $\frac{1}{2}\cdot -\frac{7}{13} = -\frac{7}{26}$, and $\left(-\frac{7}{26}\right)^2 = \mathbf{\frac{49}{676}}$

3. a. 2 b. 1 c. 2 d. 0

4. a. $(n+5)^2$ b. $(x-9)^2$ c. $\left(a+\frac{5}{2}\right)^2$

d. $\left(x-\frac{2}{3}\right)^2$ e. $\left(y+\frac{3}{7}\right)^2$ f. $\left(t-\frac{1}{10}\right)^2$

5. $\frac{49}{324}$

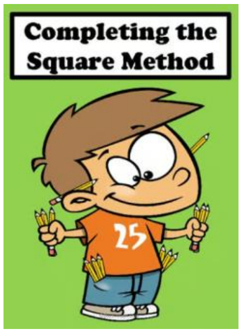
Education is not

a preparation for life;

education is life itself.

John Dewey

CH N – COMPLETING THE SQUARE



If you understand all the concepts in the chapter *Preparing for Completing the Square*, you're ready to tackle this chapter.

❑ *THE FIVE STEPS IN COMPLETING THE SQUARE*

We're now ready to combine many of our algebra skills into solving non-factorable quadratic equations, like $x^2 + 3x + 1 = 0$.

We start by assuming that the quadratic equation is in standard form. If it's not, we know we can always move things around to convert it to standard form:

$$ax^2 + bx + c = 0 \qquad \textbf{(where } \boldsymbol{a \neq 0}\textbf{)}$$

1. Make sure that the leading coefficient (the *a*) is 1. If it's not, divide each side of the equation (all terms) by *a*.
2. Move the constant to the other side of the equation.
3. Compute the "magic number" and add it to both sides of the equation. This step "*Completes the Square*."
4. Factor the left side, and then simplify the right side.
5. Solve the resulting equation by taking square roots, remembering that every positive number has two square roots (the Square Root Theorem).

Note: Steps 1 and 2 can be done in either order.

☐ EXAMPLES OF COMPLETING THE SQUARE

EXAMPLE 1: **Solve by Completing the Square:**

$$\mathbf{x^2 + 8x - 20 = 0}$$

Solution: Even though this quadratic equation is factorable, we'll solve it by completing the square — a technique that will also work on problems that aren't factorable, as well as allow us to solve some problems with circles.

Step 1:

We must ensure that the leading coefficient (the a) is 1. It already is 1 ($x^2 = 1x^2$), so step 1 is done, and the equation remains the same:

$$x^2 + 8x - 20 = 0$$

Step 2:

Move the constant (the –20) to the right side of the equation by adding 20 to each side of the equation:

$$x^2 + 8x = 20$$

Step 3:

Now we calculate the *magic number*:

a) Calculate half of 8: $\frac{1}{2}(8) = 4$

b) Square that 4; the magic number is **16**.

Add the magic number to each side of the equation:

$$x^2 + 8x + \boxed{16} = 20 + \boxed{16}$$

Step 4:

Factor the left side and simplify the right side:

$$(x + 4)^2 = 36$$

Step 5:

Solve by taking square roots:

$$x + 4 = \pm\sqrt{36} \qquad \text{(36 has } \underline{\textbf{two}} \text{ square roots)}$$

$$x + 4 = \pm 6 \qquad \text{(simplify the radical)}$$

$$x = -4 \pm 6 \qquad \text{(subtract 4 from both sides)}$$

Using the plus sign ⇒ $x = -4 + 6 = 2$

Using the minus sign ⇒ $x = -4 - 6 = -10$

We have now solved our first equation by completing the square, and its solutions are

$$\boxed{x = 2 \text{ or } -10}$$

Note that we could have solved the quadratic equation by factoring. No matter the method, we would get the same solutions.

EXAMPLE 2: **Solve by Completing the Square:**

$$\mathbf{3x^2 - 5x + 1 = 0}$$

Solution: We proceed as we did above with the 5-step plan.

Step 1:

The first requirement for completing the square is a leading coefficient of 1. Since the leading coefficient in this problem is 3, we will have to divide both sides of the equation by 3:

$$\frac{3x^2 - 5x + 1}{3} = \frac{0}{3}$$

or, $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ (divide all terms by 3)

Step 2:

Move the constant to the right side, resulting in

$$x^2 - \frac{5}{3}x = -\frac{1}{3} \qquad \text{(subtract } \tfrac{1}{3} \text{ from both sides)}$$

Step 3:

It's now time for the magic number, the number that will complete the square. We calculate half of $-\frac{5}{3}$, square that result, and we'll have the "magic number" that will be added to each side of the equation.

Magic Number Calculation:

$$\frac{1}{2}\left(-\frac{5}{3}\right) = -\frac{5}{6}, \text{ and then } \left(-\frac{5}{6}\right)^2 = \mathbf{\frac{25}{36}}$$

Add this number to each side of the equation:

$$x^2 - \frac{5}{3}x + \boxed{\frac{25}{36}} = -\frac{1}{3} + \boxed{\frac{25}{36}}$$

Step 4:

Factoring the left side and adding the fractions on the right side gives us:

$$\left(x - \frac{5}{6}\right)^2 = \frac{13}{36} \qquad \left[-\frac{1}{3} + \frac{25}{36} = -\frac{12}{36} + \frac{25}{36} = \frac{13}{36}\right]$$

Step 5:

Now we take the square root of each side of the equation, remembering that the right side has **two** square roots:

$$x - \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

Next, we isolate the x by adding $\frac{5}{6}$ to each side of the equation:

$$x = \frac{5}{6} \pm \sqrt{\frac{13}{36}}$$

Split the radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{\sqrt{36}} \qquad \left[\text{from the rule: } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\right]$$

And simplify the bottom radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

Combining the fractions into a single fraction (the LCD is 6) produces the final answer:

$$\boxed{x = \frac{5 \pm \sqrt{13}}{6}}$$

Our quadratic has two solutions.

Homework

1. At what point in Completing the Square could you determine that the quadratic equation you're trying to solve has NO solution?

2. Solve each quadratic equation by Completing the Square:

a. $y^2 - 6y + 5 = 0$	b. $x^2 + 13x + 30 = 0$
c. $z^2 + 5z - 14 = 0$	d. $2n^2 - n - 1 = 0$
e. $12t^2 - 5t - 3 = 0$	f. $10a^2 + 7a + 1 = 0$
g. $x^2 + 25 = 10x$	h. $4u^2 + 20u + 25 = 0$
i. $w^2 = -w - 5$	j. $2h^2 + 1 = h$

3. Solve each quadratic equation by Completing the Square:

a. $x^2 + 3x + 1 = 0$

b. $y^2 - 4y + 2 = 0$

c. $2a^2 + 6a + 3 = 0$

d. $n^2 + 8n - 2 = 0$

e. $3u^2 - 4u - 2 = 0$

f. $t^2 + 10t + 3 = 0$

g. $5w^2 + w + 1 = 0$

h. $2x^2 = -5x - 1$

i. $g^2 = 3g - 5$

j. $3m^2 = 1 - 4m$

4. Solve each quadratic equation by Completing the Square:

a. $q^2 + 8q + 15 = 0$

b. $2x^2 - 7x + 1 = 0$

c. $3n^2 + n - 5 = 0$

d. $4a^2 - 5a = -3$

e. $5w^2 = -2 - 7w$

f. $z^2 + 1 = 2z$

Solutions

1. As soon as you notice that you've reached the square root of a negative number, you can stop and conclude that the quadratic equation has no solution (in the real numbers).

2. a. $y = 1, 5$

b. $x = -3, -10$

c. $z = 2, -7$

d. $n = 1, -\frac{1}{2}$

e. $t = \frac{3}{4}, -\frac{1}{3}$

f. $a = -\frac{1}{2}, -\frac{1}{5}$

g. $x = 5$

h. $u = -\frac{5}{2}$

i. No solution

j. No solution

3. a. $x = \frac{-3 \pm \sqrt{5}}{2}$
b. $y = 2 \pm \sqrt{2}$
c. $a = \frac{-3 \pm \sqrt{3}}{2}$
d. $n = -4 \pm 3\sqrt{2}$
e. $u = \frac{2 \pm \sqrt{10}}{3}$
f. $t = -5 \pm \sqrt{22}$
g. No solution
h. $x = \frac{-5 \pm \sqrt{17}}{4}$
i. No solution
j. $m = \frac{-2 \pm \sqrt{7}}{3}$

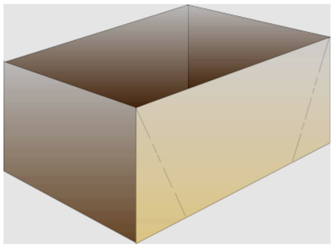
4. a. $q = -5, -3$
b. $x = \frac{7 \pm \sqrt{41}}{4}$
c. $n = \frac{-1 \pm \sqrt{61}}{6}$
d. No solution
e. $w = -\frac{2}{5}, -1$
f. $z = 1$

“The true sign of intelligence is not knowledge, but imagination.”

Albert Einstein

CH XX – CUBIC FUNCTIONS

You're given a square piece of cardboard, 8 cm on a side, and told to cut squares of equal size out of each corner and fold up the resulting flaps, all to create an open box (no top) with the *largest* possible volume. What size corners should you cut from the cardboard? Solving this problem involves a cubic function.

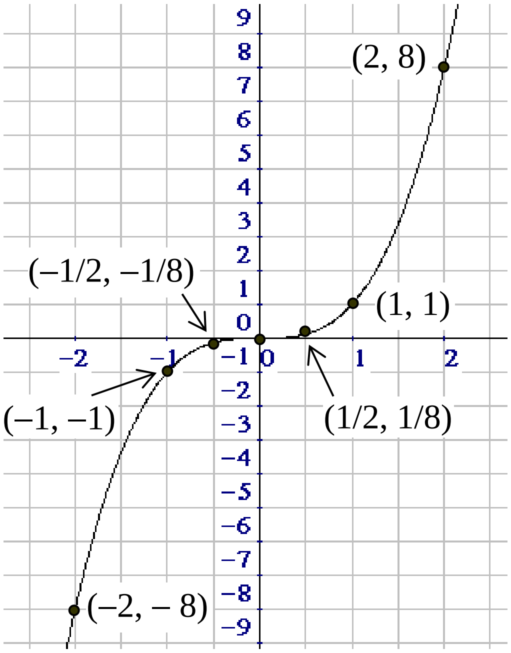


☐ *INTRODUCTORY EXAMPLE*

EXAMPLE 1: **Graph: $y = x^3$**

Solution: This is the classic cubic function — it's as simple as a cubic can be. The **domain** is $\mathbb{R}$ (since any number can be cubed without a problem). Confirm the values in the following *x*-*y* table.

x	y
-3	-27
-2	-8
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
3	27



Note that when $x = 0$, $y = 0$; and when $y = 0$, $x = 0$. Therefore, the only **intercept** is the origin. Also notice that the **range** of the function is $\mathbb{R}$.

Also, the graph of the function appears to have **origin symmetry**. To prove this, replace x with $-x$ and y with $-y$, giving

$-y = (-x)^3 \quad \Rightarrow \quad -y = -x^3 \quad \Rightarrow \quad y = x^3$, the original equation.

In addition, **these limits** should be clear:

As $x \to \infty$, $y \to \infty$ As $x \to -\infty$, $y \to -\infty$

Homework

1. a. In Example 1, we claimed that the domain is $\mathbb{R}$, so we should be able to cube any real number. Cube each of these numbers: 10, π, $\sqrt[3]{7}$, $-\sqrt[6]{11}$.

   b. It's also the case that y can be any real number. Find an x-value that will yield the given y-value: 64, $\frac{1}{27}$, -125, π, $-e$. [e is a number we'll learn about later in the course.]

2. Graph $y = -x^3$. Remember, $-x^3$ means cube x first, and then attach the minus sign.

3. Graph $f(x) = x^3 + 2$ and $g(x) = x^3 - 3$, and then explain what effect the 2 and the -3 have on the graph of $y = x^3$.

4. Graph $f(x) = (x-1)^3$ and $g(x) = (x+4)^3$, and then explain what effect the -1 and the 4 have on the graph of $y = x^3$.

5. For the function $y = x^3$, verify the limit "As $x \to \infty$, $y \to \infty$" by finding the x-value needed to guarantee that y can be made as large as one billion.

6. What is the slope of the curve $y = x^3$ at the origin?

7. This problem will prepare us for an issue in the next section. Let's solve the quadratic equation $x^2 + 7x - 18 = 0$ by factoring.

$$x^2 + 7x - 18 = 0$$
$$\Rightarrow \quad (x + 9)(x - 2) = 0$$
$$\Rightarrow \quad x + 9 = 0 \text{ or } x - 2 = 0$$
$$\Rightarrow \quad x = -9 \text{ or } x = 2$$

Notice that the solution $x = -9$ came from the factor $x + 9$, while the solution $x = 2$ came from the factor $x - 2$. So here's the question: What quadratic equation has the solutions $x = 7$ and $x = -5$? If 7 and -5 are solutions, then $x - 7$ and $x + 5$ must be factors: $(x - 7)(x + 5) = 0$, or $x^2 - 2x - 35 = 0$.

Find a quadratic equation whose solutions are given:

a. $x = 5, 10$

b. $x = -4, 9$

c. $x = -2, -11$

d. $x = 12$

e. $x = \pm 7$

f. $x = 0, 5$

☐ *INTERCEPTS AND GRAPHING*

EXAMPLE 2: **Graph: $y = x^3 - 9x$**

Solution: This is another cubic function whose **domain** is ℝ.

Let's analyze **intercepts**. If we let $x = 0$, then $y = 0$, and so the graph has a y-intercept at the origin. Setting y to 0 gives

$$0 = x^3 - 9x = x(x^2 - 9) = x(x + 3)(x - 3),$$

whose solutions are $x = 0$, $x = -3$, and $x = 3$. Thus the x-intercepts are the points (0, 0), (–3, 0), and (3, 0).

Let's check for **symmetry** — we'll try y-axis symmetry first. Replace x with $-x$ and we get $y = (-x)^3 - 9(-x) = -x^3 + 9x$. This is not the same formula as the original, so we do not have y-axis symmetry. But you can check that if we replace x with $-x$ and y

with $-y$, then we <u>do</u> get the same equation, and we can conclude that the graph has origin symmetry.

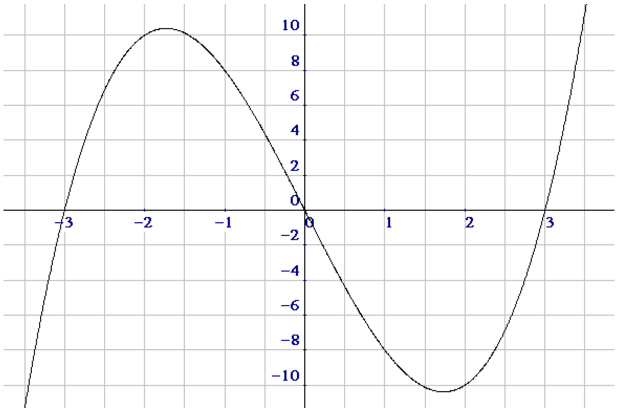
A few more calculations will give us the points

$$(-4, -28) \quad (-2, 10) \quad (-1, 8) \quad (1, -8) \quad (2, -10) \quad (4, 28).$$

Now let x be really big and really small, and you should be convinced of **these limits**:

$$\text{As } x \to \infty, y \to \infty \qquad \text{As } x \to -\infty, y \to -\infty$$

These points, along with the intercepts, the origin symmetry, and the limits, lead us to the following graph:



<u>EXAMPLE 3:</u> **Find all the x-intercepts of**

$$y = x^3 - x^2 - 14x + 24.$$

Hint: One of them is (2, 0).

<u>Solution:</u> Why the hint? To find the x-intercepts of any graph, we set $y = 0$. This yields the equation

$$x^3 - x^2 - 14x + 24 = 0$$

We can't use the Quadratic Formula since we have a cubic equation, not a quadratic one. That leaves factoring as the only viable technique, but factoring a cubic is difficult.

It's time to take advantage of the hint: (2, 0) is one of the x-intercepts. This means that the point (2, 0) is on the graph of the cubic function, which implies that the coordinates $x = 2$ and $y = 0$ must satisfy the equation of the cubic function. Let's verify this:

$$\begin{aligned} & \mathbf{0} = \mathbf{2}^3 - \mathbf{2}^2 - 14(\mathbf{2}) + 24 \\ \Rightarrow \quad & 0 = 8 - 4 - 28 + 24 \\ \Rightarrow \quad & 0 = 0 \quad \checkmark \end{aligned}$$

Now back to the cubic equation above:

$$x^3 - x^2 - 14x + 24 = 0$$

Here's the clue we need to solve this equation: Since $x = 2$ is one of the solutions of this equation (verified above), then $x - 2$ must be one factor of the left side of the equation. [See the previous homework problem.]

To find the other factors, we'll divide $x^3 - x^2 - 14x + 24$ by $x - 2$, the details of which are left to you.

$$\begin{array}{r|l} & x^2 + x - 12 \\ \cline{2-2} x - 2 & x^3 - x^2 - 14x + 24 \end{array}$$

Since the quotient is $x^2 + x - 12$ and the divisor is $x - 2$, we can write

$$x^3 - x^2 - 14x + 24 = (x^2 + x - 12)(x - 2)$$

Factoring the quadratic gives us the complete factorization. Here's the process from the beginning:

$$\begin{aligned} & x^3 - x^2 - 14x + 24 = 0 && \text{(the equation we're trying to solve)} \\ \Rightarrow \quad & (x^2 + x - 12)(x - 2) = 0 && \text{(from the long division)} \\ \Rightarrow \quad & (x + 4)(x - 3)(x - 2) = 0 && \text{(factor the quadratic)} \\ \Rightarrow \quad & x = -4 \text{ or } x = 3 \text{ or } x = 2 && \text{(set each factor to 0)} \end{aligned}$$

And we thus have all three x-intercepts:

$$\boxed{(-4, 0) \quad (3, 0) \quad (2, 0)}$$

Can you see that the y-intercept is (0, 24)?

EXAMPLE 4: **Consider the cubic function $y = x^3 + 2x^2 - x - 2$. Use the fact that one of the x-intercepts is (–1, 0) to find all the intercepts, and then graph the function.**

Solution: First, the **domain** of the function is $\mathbb{R}$. Second, as in the previous example, we realize that an x-intercept of (–1, 0) means that $x + 1$ is one factor of the cubic. Dividing $x + 1$ into $x^3 + 2x^2 - x - 2$ gives a quotient of $x^2 + x - 2$, which itself factors into $(x + 2)(x - 1)$. The complete factorization of the cubic function $y = x^3 + 2x^2 - x - 2$ is

$$y = (x + 1)(x + 2)(x - 1).$$

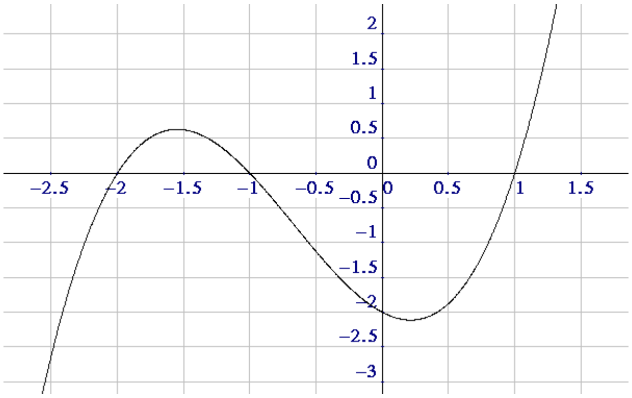
Setting y to 0 gives the following three **x-intercepts**:

(–1, 0) (–2, 0) (1, 0)

Setting x to 0 gives a **y-intercept** of (0, –2). You can verify (via calculator) that these points are on the graph, too:

(–2.25, –1.015625) (–1.5, 0.625) (0.25, –2.109375)

If you plot the three x-intercepts, the y-intercept, and the three points above, you'll get a graph like the following:



The graph does not seem to have any **symmetries**. Also, we note **these two limits**: As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

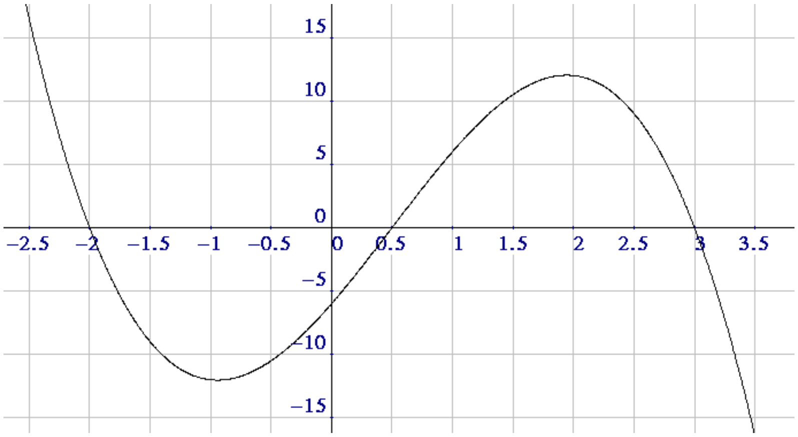
Postscript: It's impossible to accurately determine the *maximum point* (the top of the hill) in the second quadrant, or the *minimum point* (the bottom of the valley) in the fourth quadrant. Calculus is the subject where we find the tools needed to determine these two *extreme points* precisely.

EXAMPLE 5: **Find the intercepts and then sketch the graph of $y = -2x^3 + 3x^2 + 11x - 6$. Hint: One of the x-intercepts is (3, 0).**

Solution: An x-intercept of (3, 0) implies that $x - 3$ is one factor of the cubic. Dividing the cubic by $x - 3$ yields a quotient of $-2x^2 - 3x + 2$, which factors into $-(2x - 1)(x + 2)$. When this product is set to 0, we find that two additional x-intercepts are $(\frac{1}{2}, 0)$ and $(-2, 0)$. When we throw in the given x-intercept, and then calculate the y-intercept, we get four intercepts:

$$(-2, 0) \qquad (\tfrac{1}{2}, 0) \qquad (3, 0) \qquad (0, -6)$$

With these four intercepts and a few other points which I'll leave for you to plot, we get the following graph:



The limits for this graph are different from the previous examples: As $x \to \infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to \infty$. Do you know what it is about the cubic equation which produces these limits?

Homework

8. Graph $y = x^3 - 4x$.

9. Find all the x-intercepts of $y = x^3 - 4x^2 - 7x + 10$. Hint: One of them is (1, 0).

10. In Example 4, perform the long division to verify the factorization of the cubic. Now verify the calculations of the three additional points on the graph. Finally, without referring to the graph, prove that the graph has no symmetries.

11. Graph $y = x^3 - 3x^2 + 2x$. Label all the intercepts clearly. Estimate the maximum and minimum points on the graph.

12. Graph $y = -x^3 + 3x - 2$. Hint: One of the x-intercepts is $(-2, 0)$. Estimate the maximum and minimum points on the graph. As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

13. Graph $y = x^3 - 2x^2 - 5x + 6$. Hint: $(-2, 0)$ is an x-intercept.

14. Graph $y = -x^3 - 4x^2 - 4x$.
As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

☐ *AN APPLICATION OF CUBIC FUNCTIONS*

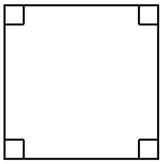
Remember the box question described in this chapter's Introduction? Now we're ready to answer that query. You may recall from Gertrude's pigpen problem that a given amount of fence could produce different areas, depending on what dimensions we chose for the rectangle. The same concept applies to the box problem. We'll begin with two specific scenarios which should convince you that we could get different volumes with the same square piece of cardboard.

EXAMPLE 6: **Start with a cardboard square 8 cm by 8 cm. First cut out 1-cm squares from the corners and fold up the flaps to create an open box, and compute its volume. Then, starting with the original 8-cm square, cut out 2-cm squares from the corners and compute the volume of the resulting box. Prove that the two boxes have different volumes.**

Solution: The volume of a box with dimensions l, w, and h is given by

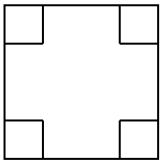
$$V = lwh$$

1-cm corners removed leaves a box 6 cm on each side and 1 cm high



$$\begin{aligned} V &= 6\text{ cm} \times 6\text{ cm} \times 1\text{ cm} \\ &= \mathbf{36\ cm^3} \end{aligned}$$

2-cm corners removed leaves a box 4 cm on each side and 2 cm high

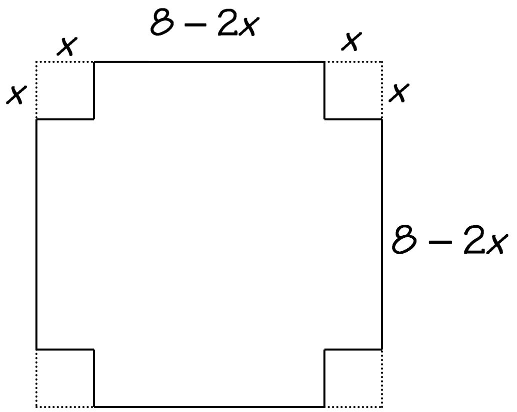


$$\begin{aligned} V &= 4\text{ cm} \times 4\text{ cm} \times 2\text{ cm} \\ &= \mathbf{32\ cm^3} \end{aligned}$$

Therefore, the boxes have different volumes, even though each box was created from identical 8-cm squares of cardboard. The next example will ask us to find the size of the cut-out corner that will produce the *maximum* volume. After all, given an 8-cm square of cardboard, we might as well get the most volume that we possibly can from the box.

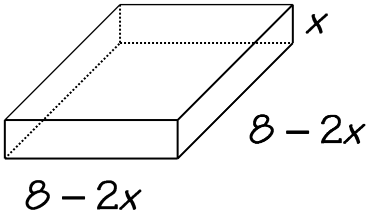
EXAMPLE 7: **An 8-cm square piece of cardboard is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume? What will the maximum volume be?**

Solution: Here's the plan of attack. We'll sketch the piece of cardboard with the corners removed; we'll assume that each square corner removed has dimensions of x cm by x cm. Then we'll write an expression that represents the length and width of the box which results from folding up the flaps. Since the height of the box is x cm, we'll be able to create a volume formula from the length, the width, and the height. Last, we'll graph the resulting function and estimate the value of x which produces the maximum volume.



The following are the dimensions of the box when the flaps are folded up:

length = $8 - 2x$
width = $8 - 2x$
height = x



And so the volume has the formula:

$$\begin{aligned} V &= lwh \\ &= (8-2x)(8-2x)x \\ &= (64-32x+4x^2)x \\ &= 4x^3-32x^2+64x \end{aligned}$$

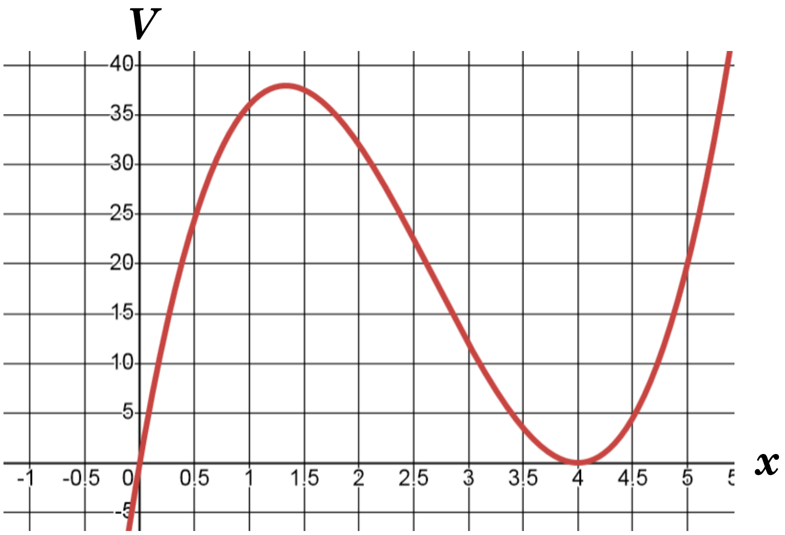
Therefore, the function we're trying to maximize is given by

$$V = 4x^3 - 32x^2 + 64x$$

Factoring produces

$$V = 4x(x-4)^2$$

which yields two x-intercepts, (0, 0) and (4, 0). Clearly we could plot lots of other points to get a decent graph, but let's just cut to the chase with the following graph:



Before we get to the goal of this problem, let's notice what the graph is telling us. It says that when $x = 0$, $V = 0$ (i.e., the graph passes through the origin). This makes sense because if we don't cut out any corners, we can't fold up the flaps, we get no box, and so obviously the volume is 0. By the same token, if we cut out flaps that are 4 cm long, there's no base left to the box, since it's only 8 cm to begin with. The graph indicates this situation at the point (4, 0).

Now to find the size of the squares to cut out and the maximum volume: Look at the top of the mountain — this point represents maximum volume. Going straight down to the x-axis, we estimate the x-coordinate to be about 1.3 cm. Now move from the top of the mountain to the left, and the volume appears to be about 38 cm^3. In summary,

Cutting a 1.3 cm square from each corner will produce a box with volume 38 cm^3.

Homework

15. Each of the following numbers is the length of the side of a piece of square cardboard. The cardboard square is to be made into an open box by cutting squares from the corners and folding up the flaps. What size square should be cut from each corner to achieve a box of maximum volume? Note that the solutions are approximations only. As long as your answers are in the ballpark (and you really understand what you're doing!), consider yourself correct.

a. 2 b. 3 c. 4 d. 6

Review Problems

16. Let $y = 2x^3 - 4x^2 + 9$.

a. Is it a function? Why?

b. Why is it cubic?

c. What's its domain?

d. Find the y-intercept.

e. What would you do to find the x-intercepts?

17. How does the graph of $g(x) = (x - 2)^3 + 7$ compare to the graph of $f(x) = x^3$?

18. Prove that the graph of $y = 2x^3 + 7x$ has origin symmetry.

19. Let $f(x) = 3x^3 + 4x - 7$ and $g(x) = -\pi x^3 + x^2 + 100$.

    a. As $x \to \infty$, $f(x) \to$ _____ b. As $x \to -\infty$, $f(x) \to$ _____

    c. As $x \to \infty$, $g(x) \to$ _____ d. As $x \to -\infty$, $g(x) \to$ _____

20. If one of the x-intercepts of the cubic graph $y = x^3 + 3x^2 - 61x - 63$ is (7, 0), find the other two intercepts.

21. Graph $y = -x^3 + 2x$. Be sure to discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

22. Graph $y = x^3 + 3x^2 - x - 3$. Hint: (1, 0) is one of the intercepts. Discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

23. A 5-cm square piece of aluminum is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume?

24. True/False:

    a. Both the domain and range of the function $y = x^3$ are $\mathbb{R}$.

    b. The slope of the curve $y = x^3$ at the origin is about 1.

    c. The curve $y = x^3$ never touches the 4th quadrant.

    d. For the function $y = x^3$, as $x \to -\infty$, $y \to 0$.

    e. The graphs of $f(x) = (-x)^3$ and $g(x) = -x^3$ are identical.

    f. The graph of $y = x^3 - 16x$ has three x-intercepts.

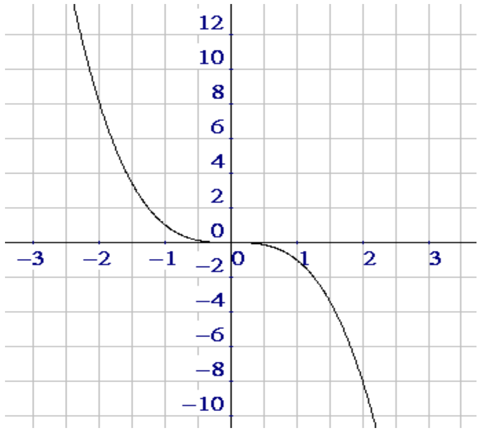
    g. The graph of $y = x^3 - 25x$ has its minimum point in Quadrant II.

    h. If you know one of the x-intercepts of a cubic function, long division will help you find the others.

Solutions

1. a. $1000,\ \pi^3,\ 7,\ -\sqrt{11}$ b. $4,\ \frac{1}{3},\ -5,\ \sqrt[3]{\pi},\ \sqrt[3]{-e}$

2.



3. The graph of f is x^3 shifted up 2 units.
The graph of g is x^3 shifted down 3 units.

4. The graph of f is x^3 shifted right 1 unit.
The graph of g is x^3 shifted left 4 units.

5. Make $x \geq 1000$ and y will be $\geq 1{,}000{,}000{,}000$.

6. I'd like to know your opinion.

7. a. $(x-5)(x-10) = 0 \Rightarrow x^2 - 15x + 50 = 0$

b. $x^2 - 5x - 36 = 0$

c. $x^2 + 13x + 22 = 0$

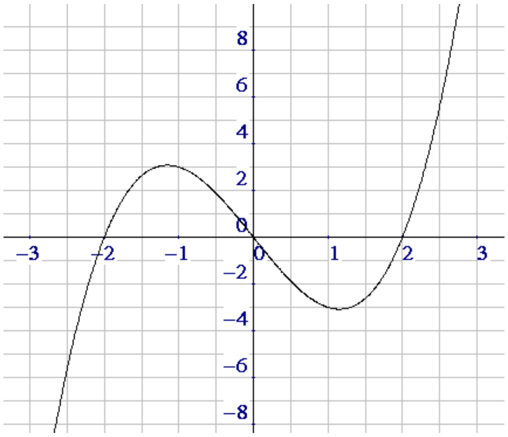
d. Only one solution is given, but we're required to provide a quadratic equation; so we must use the solution $x = 12$ twice:
$(x-12)(x-12) = 0$, or $x^2 - 24x + 144 = 0$

e. $x^2 - 49 = 0$

Any of the quadratic equations here could be multiplied by any nonzero constant and still have the same solutions. I've chosen the simplest answers.

f. $x(x-5) = 0 \quad \Rightarrow \quad x^2 - 5x = 0$

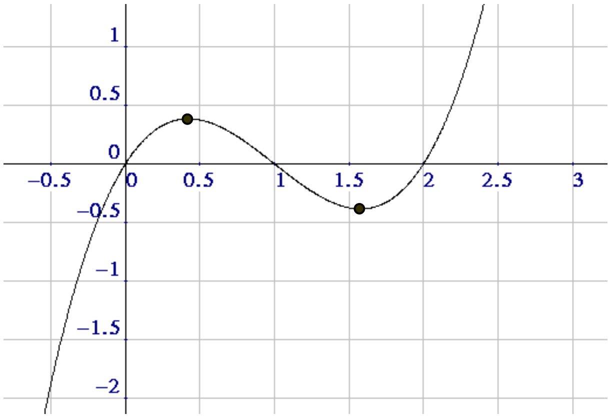
8.



9. Divide $x - 1$ into $x^3 - 4x^2 - 7x + 10$; the quotient is $x^2 - 3x - 10$. The complete factorization is $(x-1)(x-5)(x+2)$, and therefore the x-intercepts are (1, 0), (5, 0) and (–2, 0).

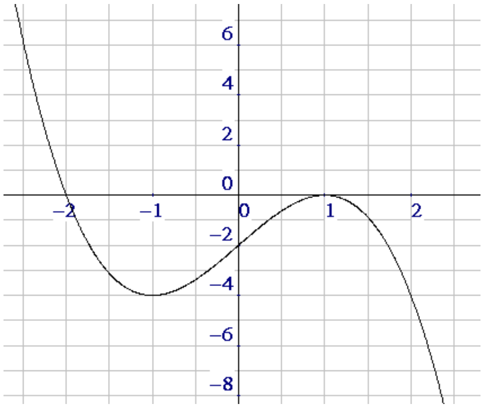
10. For the symmetries, use the litmus tests; e.g., to test for y-axis symmetry, substitute $-x$ for x and show that you do not get the same equation.

11.



A maximum point appears roughly at (.4, .4), and a minimum point near (1.6, –.4).

12.

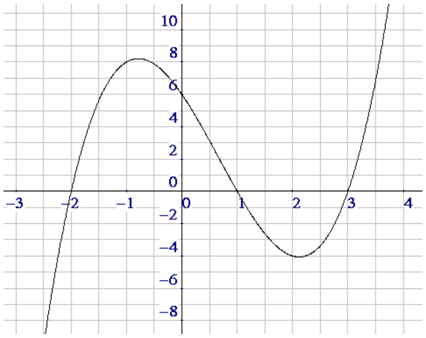


max at (1, 0)

min at (–1, –4)

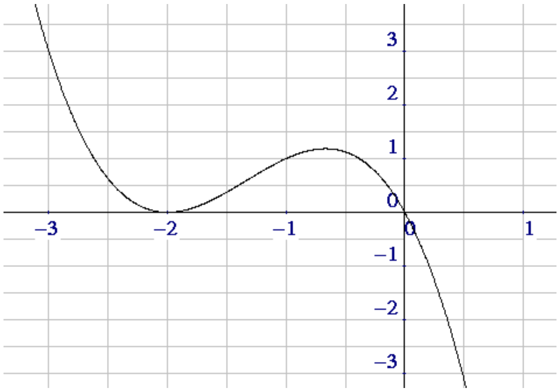
Limits: –∞; ∞

13.



Make sure your four intercepts match those on the graph.

14.



Limits: –∞; ∞

15. a. 0.3 b. 0.5 c. 0.7 d. 1

16. a. Yes; Given any input (an x), there's only one output (a y).

b. Because the highest exponent is 3.

c. ℝ, since there's no real number that could possibly cause any problems.

d. Setting $x = 0$ gives $y = 9$. Therefore, the y-intercept is $(0, 9)$.

e. Set $y = 0$, then solve the resulting equation for x.

17. Compared to f, the graph of g is 2 units to the right and 7 units up.

18. We replace x with $-x$ and y with $-y$ at the same time:

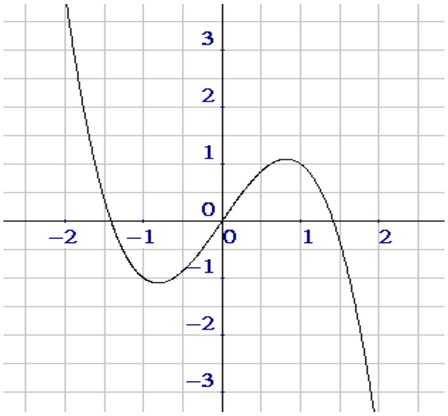
$$\begin{aligned} & -y = 2(-x)^3 + 7(-x) \\ \Rightarrow\quad & -y = 2(-x^3) + (-7x) \\ \Rightarrow\quad & -y = -2x^3 - 7x \\ \Rightarrow\quad & y = 2x^3 + 7x, \text{ the same as the original equation.} \end{aligned}$$

Therefore, the graph has origin symmetry.

19. a. ∞ b. $-\infty$ c. $-\infty$ d. ∞

20. $(-1, 0)$ and $(-9, 0)$

21.



Origin symmetry

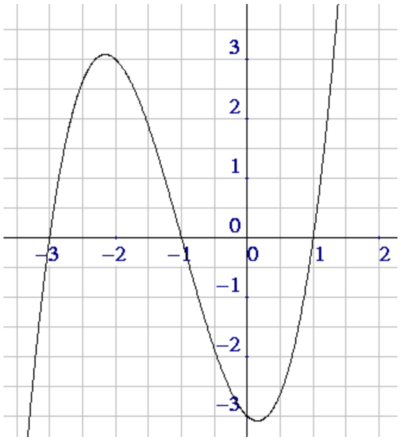
x-int: $(0, 0)$ and $(\pm\sqrt{2}, 0)$

y-int: $(0, 0)$

Domain = ℝ

Limits: $-\infty$; ∞

22.



No symmetry

x-int: (1, 0), (–1, 0), and (–3, 0)

y-int: (0, –3)

Domain = $\mathbb{R}$

Limits: ∞; $-\infty$

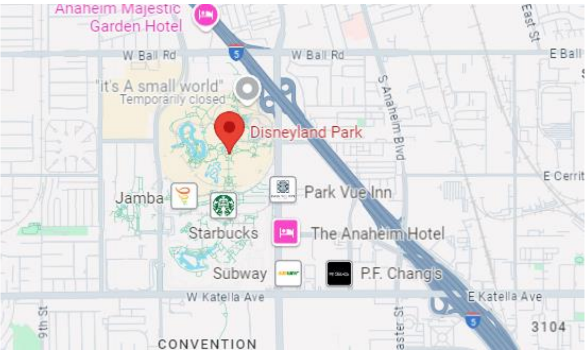
23. approximately 0.8

24. a. T b. F c. T d. F e. T f. T g. F h. T

Give a man a fish
and you feed him for a day.
Teach a man to fish
and you feed him for a lifetime.

Chinese Proverb

CH NN –DISTANCE ON THE LINE AND IN THE PLANE



One important factor when visiting Disneyland is the **distance** between your home and the theme park. It may determine your mode of transportation, the travel time, and thus the total cost of the trip.

You might want to review the chapter *Absolute Value, the Basics* to help you understand this chapter.

☐ *DISTANCE ON A LINE*

Distance is never negative.

Our plan now is to create a formula that will give us the ***distance*** between two points on a line. Consider the two points 10 and 17 on a line. Is it pretty clear that the distance between them is 7? If it's not really obvious, you can simply subtract the smaller number from the larger one:

$$\textbf{\textit{larger}} - \textbf{\textit{smaller}} = 17 - 10 = \mathbf{7} \ \checkmark$$

This formula (*larger* – *smaller*) works perfectly for any two numbers on a line:

The distance between –7 and 5 = 5 – (–7) = 5 + 7 = **12**.

[Note that 5 is larger than –7.]

The distance between –9 and –20 = –9 – (–20) = –9 + 20 = **11**.

[Note that –9 is larger than –20.]

The distance between 12 and 12 = 12 – 12 = **0**.

[Note that it doesn't matter which you choose as the larger number.]

NOTE: Subtracting in the wrong order (*smaller – larger*) is catastrophic. For example, if we try to find the distance between 10 and 17 like this:

$$10 - 17 = -7$$

we get a negative distance — quite forbidden!

Our way out of this mess is to come up with a *formula* for the distance between any two points on a line. In other words, we need a formula for the distance between the numbers ***a*** and ***b*** on a line, when we may not know which one of them, *a* or *b*, is the larger one. [See the NOTE above.]

Here's the secret: Use ***absolute value***. Then, if we were to "accidentally" subtract in the wrong direction — and end up with a negative distance (which DOESN'T EXIST) — the absolute value will automatically convert that negative number into a positive number.

Moreover, if we properly subtract *larger – smaller*, we'll get a positive number, whose absolute value will be just that number. And if the two numbers are the same, the absolute value of 0 is still 0. No matter the sizes of the two numbers, subtracting them (regardless of which one's bigger, or even if they're the same) — and then applying the absolute value to the difference — works every time.

So, in short, if ***a*** and ***b*** are any two numbers on the number line, we don't even have to worry about which one is bigger. We calculate the distance between them by using the following formula:

$$\boxed{d = |a - b|}$$

The distance between two points on the line is the absolute value of their difference.

Four Examples:

A. The distance between 7 and 3 is $|7-3| = |4| = \mathbf{4}$

B. The distance between 10 and 25 is $|10-25| = |-15| = \mathbf{15}$

C. The distance between π and π is $|\pi-\pi| = |0| = \mathbf{0}$

D. The distance between –17 and –5 is

$$|-17-(-5)| = |-17+5| = |-12| = \mathbf{12}$$

Homework

1. Find the **distance** between the given pair of points on the number line, using the absolute-value formula, and show each step:

a. 7 and 2 b. –2 and 9 c. –3 and –3

d. 99 and –99 e. –5 and –13 f. –20 and –4

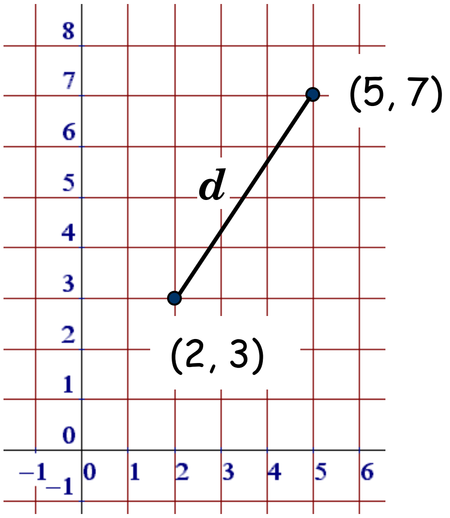
☐ *DISTANCE IN THE PLANE [USING A TRIANGLE]*

Assuming that you know how to plot points in the plane and remember the Pythagorean Theorem, we can tackle the question:

*How do we find the **distance** between two points in the **plane**?*
If the Earth were flat, it would be like asking how far apart two cities are if we know the latitude and longitude of each city.

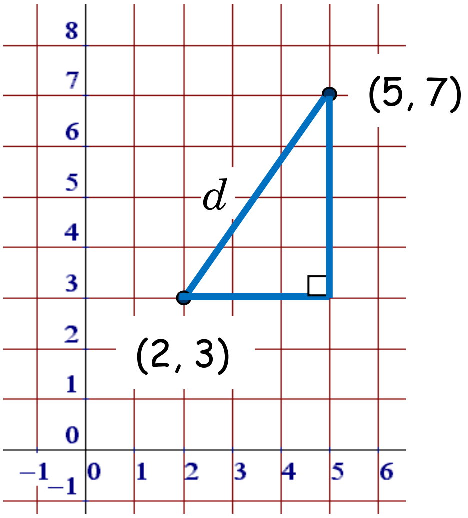
EXAMPLE 1: **Find the distance between the points (2, 3) and (5, 7) in the plane.**

Solution: Let's draw a picture and see what we can see. We'll plot the two given points and connect them with a straight line segment. The distance between the two points, which we'll call ***d***, is simply the length of that line segment.



How far is it between the two points (2, 3) and (5, 7)? Equivalently, what is the length (*d*) of the line segment connecting the two points?

Now what do we do? Well, here comes the interesting part. If we're creative enough, we might see that the segment connecting the two points can be thought of as the *hypotenuse* of a right triangle — as long as we sketch in a pair of legs to create such a triangle. Let's do that:



We've created a right triangle whose legs have lengths 3 and 4, and whose hypotenuse has a length equal to the distance between the two given points.

Sure enough, we've

constructed a right triangle where d is the length of the hypotenuse. If we can determine the lengths of the legs, then we can use the Pythagorean Theorem to find the length of the hypotenuse. By counting squares along the base of the triangle, we see that one leg is 3. Similarly, the other leg (the height) is 4. Since the square of the hypotenuse is equal to the sum of the squares of the legs, we can write the equation

$$
\begin{aligned}
& d^2 = 3^2 + 4^2 && \text{(Pythagorean Theorem: } \text{hyp}^2 = \text{leg}^2 + \text{leg}^2\text{)} \\
\Rightarrow\quad & d^2 = 9 + 16 && \text{(square the legs)} \\
\Rightarrow\quad & d^2 = 25 && \text{(add)} \\
\Rightarrow\quad & d = 5 && \text{(since } \sqrt{25} = 5\text{)}
\end{aligned}
$$

You may notice that $d = -5$ also satisfies the equation $d^2 = 25$, since $(-5)^2 = 25$; but does a negative value of d make sense? No, because distance can never be negative; so we conclude that

$$\boxed{\text{The distance between the two points is } \mathbf{5}}$$

Homework

2. By plotting the two given points in the plane and using the Pythagorean Theorem, find the **distance** between the points:

a. (1, 1) and (4, 5)

b. (2, –3) and (6, –6)

c. (–3, 5) and (2, –7)

d. (–4, –5) and (1, 7)

e. (–5, 0) and (1, 8)

f. the origin and (6, 8)

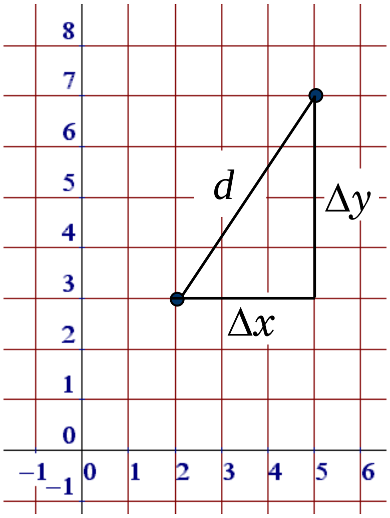
g. the origin and (–5, –12)

h. (2, 5) and (2, –1)

i. (–3, 4) and (2, 4)

j. (π, 99) and (π, 99)

☐ DISTANCE IN THE PLANE [USING A FORMULA]



To find the distance between two points in the plane, we've learned to create a right triangle so that the hypotenuse is the distance between the points.

Notice that the bottom leg is the change in x (Δx), while the vertical leg is the change in y (Δy). Thus, by employing the Pythagorean Theorem, we know that $d^2 = (\Delta x)^2 + (\Delta y)^2$. Solving this equation for d (and ignoring the negative square root), results in our Distance Formula:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

EXAMPLE 2: **Find the distance between the points (2, –8) and (12, –14) in the plane.**

Solution: Using the formula we just created (and with no reference to a graph), we can calculate the following:

$$\mathbf{\Delta x} = 2 - 12 = \mathbf{-10}$$

$$\mathbf{\Delta y} = -8 - (-14) = -8 + 14 = \mathbf{6}$$

So the distance between the points is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Note: If we calculate Δx or Δy by subtracting in the reverse order, it doesn't matter in the distance formula, since these changes are being squared anyway.

$$\Rightarrow \quad d = \sqrt{(-10)^2 + (6)^2}$$

$$\Rightarrow \quad d = \sqrt{100 + 36}$$

$$\Rightarrow \quad d = \sqrt{136}$$

$$\Rightarrow \quad d = \boxed{2\sqrt{34}}$$

☐ *THE CLASSIC DISTANCE FORMULA*

An alternative formula, commonly found in books, is to realize that Δx is the difference of the x-values, and Δy is the difference of the y-values, so that the distance between the points (x_1, y_1) and (x_2, y_2) is

$$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Homework

3. Use either the Distance Formula with the Δx and Δy in it or the classic Distance Formula to find the **distance** between the given pairs of points:

a. (1, 1) and (4, 6)

b. (–2, –3) and (6, –6)

c. (–3, 5) and (0, –7)

d. (–4, 5) and (1, 7)

e. (–9, 0) and (1, 8)

f. the origin and (6, 11)

g. the origin and (–4, –10)

h. (2, 5) and (2, –1)

i. (–3, 4) and (2, 4)

j. $(\pi, \sqrt{2})$ and $(\pi, \sqrt{2})$

❑ To ∞ and Beyond

Find the **distance** in 3-space between the points (1, –5, 8) and (–4, 9, 9).

Solutions

1. a. $|7-2| = |5| = 5$ b. $|-2-9| = |-11| = 11$

c. $|-3-(-3)| = |-3+3| = |0| = 0$

d. $|99-(-99)| = |99+99| = |198| = 198$

e. 8 f. 16

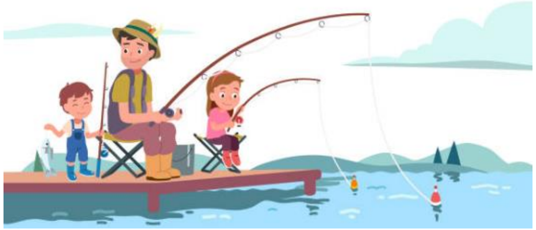
2. a. 5 b. 5 c. 13 d. 13 e. 10

f. 10 g. 13 h. 6 i. 5 j. 0

3. a. $\sqrt{34}$ b. $\sqrt{73}$ c. $3\sqrt{17}$ d. $\sqrt{29}$ e. $2\sqrt{41}$

f. $\sqrt{157}$ g. $2\sqrt{29}$ h. 6 i. 5 j. 0

“Give a man a fish
and you feed him for a day.
Teach a man to fish
and you feed him for a lifetime.”



Chinese Proverb

CH XX – DOMAIN

Have you ever gotten an ERROR message on your calculator? You may have tried to divide by zero — or perhaps you attempted to compute the square root of a negative number. In either case, you tried to use a number outside the *domain* of the *function* you were trying to calculate.

☐ *TWO WAYS TO DESCRIBE THE DOMAIN*

The ***domain*** of a function is the set of **all inputs** to the function.

There are two ways we specify the domain of a function.

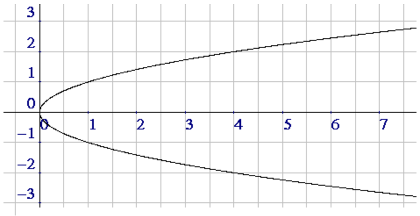
Sometimes, the domain is **explicitly** given (very common in Calculus). For instance,

> Let h be the function defined on the set of numbers [0, 3] by the formula $h(x) = x^2$.

The domain of this function is the set **[0, 3]**. Why? Because the definition says so. Thus, while $h(0) = 0$ and $h(2.5) = 6.25$ and $h(3) = 9$, the fact is that $h(-1)$ is undefined and $h(4)$ is also undefined. It's not that –1 and 4 can't be squared — they simply are not in the explicitly given domain, [0, 3].

Other times, the domain is not explicitly given, so we agree to use the ***natural domain***. In this scenario, the function's domain is every real number that's legal to use in the formula. For example, the function $g(x) = x^2$ would have a domain of $\mathbb{R}$, because any real number can be squared. But in the formula $y = \dfrac{3}{x-7}$, x can be any real number except 7 (otherwise, we're dividing by 0). Therefore, the domain is $\mathbb{R} - \{7\}$.

The concept of domain also applies to formulas that are not functions. Consider the formula $x = y^2$. We know this is not a function because an input of $x = 25$, for instance, produces two outputs, $y = \pm5$. Nevertheless, we can ask what inputs are allowed. Since x is the square of y, it should be clear that x must be greater than or equal to zero. That is, the domain is **[0, ∞)**.



The non-function $x = y^2$ has domain [0, ∞)

Homework

1. Consider the function $f(x) = \dfrac{6x-6}{\sqrt{2x-8}}$. Calculate each functional value: a. $f(12)$ b. $f(36)$ c. $f(4.5)$ d. $f(4)$ e. $f(3)$ From these results we see that 12, 36, and 4.5 are in the _______ of the function, while 4 and 3 are not.

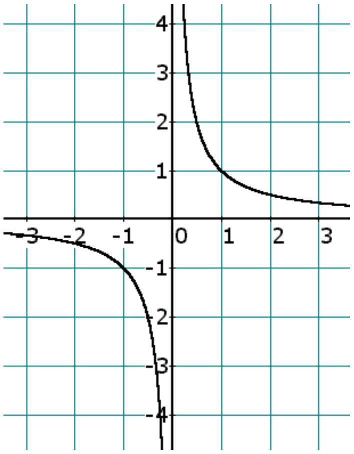
2. What is the domain of the function defined by $y = x^3$, $x \in [3, 5]$?

3. Consider the function given by the formula $g(x) = x^3 - x^2$. What is the domain of g?

❑ PRELIMINARY EXAMPLES

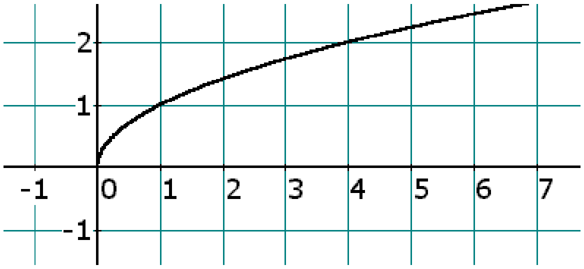
We have a pretty good notion of what a function is. There's a set of *inputs*, which are in some way associated with a set of *outputs*. Although this association is usually given by a formula (e.g., $y = x^2$), don't forget about the four quarters of the football game and the graphs in a previous chapter. Most importantly, remember that this association between the inputs and the outputs is such that *each element in the set of inputs is associated with exactly one output*. This chapter focuses on what inputs are allowed in a function.

First Example: Consider the function $y = \frac{1}{x}$. What inputs are allowed in this function? That is, what can x legally be? Well, x had better not be 0, since division by 0 is undefined. But if x is any other number, no problems will occur. So x can be any real number except 0.



The set "All real numbers except 0" can be written $\mathbb{R} - \{0\}$.

Second Example: Now consider the function $y = \sqrt{x}$. We ask the same question: What x's are allowed in this formula? In other words, what kinds of real numbers are we allowed to take the square root of? The answer is that we can take the square root of any number that is zero or greater — which is equivalent to saying that we cannot take the square root of a negative number. Thus, x can be any number greater than or equal to 0, which we also write as $\boldsymbol{x \geq 0}$, or, if you're familiar with interval notation: $\mathbf{[0, \infty)}$.



Homework

4. Let $y = 2x + 10$. Which of the following are legal values of x?

   a. 30 b. $-\pi$ c. $\sqrt{7}$ d. 0

5. Let $y = \frac{3}{x-2}$. Which of the following are legal values of x?

   a. 3 b. 2 c. π d. 0

6. Let $y = \sqrt{x}$. Which of the following are legal values of x?

   a. 7 b. 0 c. -9 d. 99

7. Let $y = \frac{10}{x^2 - 144}$. Which of the following are legal values of x?

   a. 12 b. 20 c. 0 d. -12

8. Let $y = \frac{x}{x^2 + 9}$. Which of the following are legal values of x?

   a. 3 b. 0 c. -3 d. 2

☐ *THE DOMAIN OF A FUNCTION*

The **domain** of a function is the set of all legal inputs.

We know that the ***domain*** of a function is the set of legal inputs to a function. Why is the domain a vital idea to consider? Total havoc can result when a number outside the domain is introduced into the function (like in a spreadsheet or a programming language).

For example, if our computer application tries to let $x = 0$ in the function $y = \frac{1}{x}$, we're sunk — the computer will stop execution of the program and give an error message (or worse, freeze up!). And if we allow $x = -4$ in the function $y = \sqrt{x}$, then we're really up a creek, since

$\sqrt{-4}$ is not a real number, and we're assuming that real numbers are all we have at our disposal in your course.

To review, the **domain** of a typical function of x found in an algebra or trig course is the set of all real numbers that are legal for x to be in the formula. For example, the domain of the function $y = x^2$ is $\mathbb{R}$, all the real numbers (since any real number can be squared without any serious problem). But in the formula $y = \frac{3}{x-7}$, x can be any real number except 7 (why?), and therefore the domain is $\mathbb{R} - \{7\}$, all the real numbers with 7 removed.

Homework

9. Consider the function $y = \sqrt{2x-8}$.
Calculate the functional value for each x:
a. 12 b. 36 c. 4.5 d. 4 e. 3

From these results, we see that the values 12, 36, 4.5, and 4 are in the **function's domain**, while 3 is not.

10. Let h be the function defined by $y = \frac{1}{x^2-9}$. Calculate each functional value:

x	−4	−3	−2	0	1	3	4	5
y								

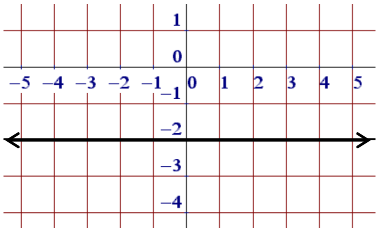
From this table of inputs and outputs, guess what the function's domain is.

11. Consider the function given by the formula $y = x^3 - x^2$. What is the domain of g?

12. What is the domain of the function $y = \frac{2}{x+3}$?

☐ FINDING THE DOMAIN

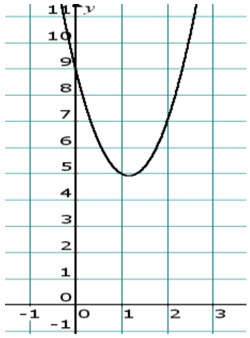
I. $y = -2$



In this equation, the x isn't even mentioned. There can, therefore, be no restrictions on x. That is, x can be any real number. Thus, the domain is

$$\boxed{\mathbb{R}}$$

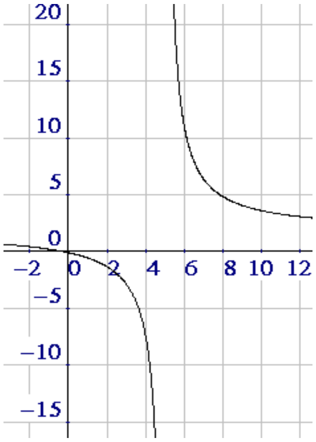
II. $y = 3x^2 - 7x + 9$



We ask ourselves: What are the legal x's? Well, x can be anything, since the operations in the formula could not possibly be a cause for concern. The domain is therefore

$$\boxed{\mathbb{R}}$$

III. $y = \dfrac{7x+2}{4x-20}$



Now we've got something interesting to look at. Question: What can go wrong in a division problem? Answer: The possibility of dividing by zero. We must ensure that x is <u>never</u> allowed to be a number that would make the denominator zero. So we find out what value(s) of x <u>would</u> make the bottom zero, and then <u>don't allow those x's to be in the domain</u>! Setting the bottom to zero gives

$$4x - 20 = 0 \longleftarrow \text{This is what we } \underline{\text{don't}} \text{ want to happen.}$$

$$\Rightarrow \quad 4x = 20$$

$$\Rightarrow \quad x = 5$$

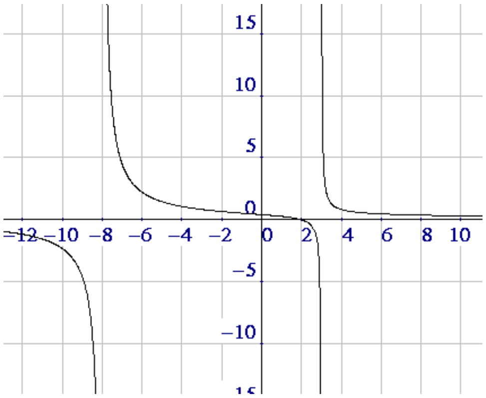
Thus, if $x = 5$, the denominator is zero, which is absolutely forbidden! So, the domain of this function is the set of all real numbers except 5:

$$\boxed{\mathbb{R} - \{5\}}$$

IV. $$y = \frac{5x - 10}{x^2 + 5x - 24}$$

As in the previous example, we must make certain that the denominator is never zero; let's see what values of x make it zero, and then exclude such values from our domain:

$$\begin{aligned} & x^2 + 5x - 24 = 0 \\ \Rightarrow \quad & (x + 8)(x - 3) = 0 \\ \Rightarrow \quad & x + 8 = 0 \quad \text{or} \quad x - 3 = 0 \\ \Rightarrow \quad & x = -8 \quad \text{or} \quad x = 3 \end{aligned}$$



We conclude that the domain is all real numbers except –8 and 3:

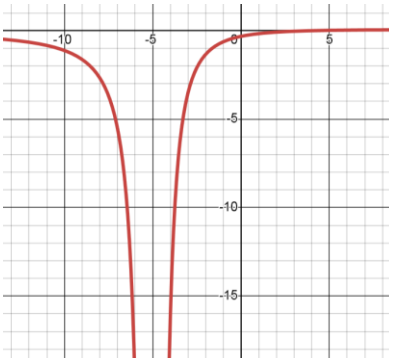
$$\boxed{\mathbb{R} - \{-8, 3\}}$$

The Quadratic Formula could have been used to solve the quadratic equation in this problem.

V. $$y = \frac{2x - 8}{x^2 + 10x + 25}$$

First, we note that it's perfectly O.K. for the numerator of a fraction to be 0, so the numerator has no role in determining the function's domain. We focus on determining what values of x would make the bottom zero, and then exclude those values from the domain:

$$\begin{aligned} & x^2 + 10x + 25 = 0 \\ \Rightarrow\quad & (x+5)(x+5) = 0 \\ \Rightarrow\quad & x + 5 = 0 \text{ or } x + 5 = 0 \\ \Rightarrow\quad & x = -5 \text{ or } x = -5 \end{aligned}$$



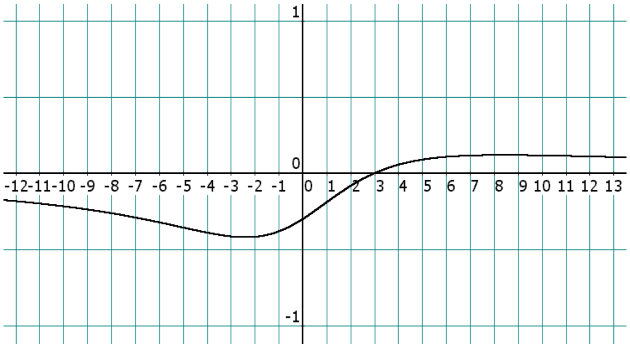
We see that the only value of x that is not allowed in the domain is $x = -5$. Hence, the domain of the function is all real numbers except –5:

$$\boxed{\mathbb{R} - \{-5\}}$$

VI. $y = \dfrac{2x-6}{x^2+20}$

Let's see what makes the denominator zero (so we can exclude it from the domain.)

$$\begin{aligned} & x^2 + 20 = 0 \\ \Rightarrow\quad & x^2 = -20 \\ \Rightarrow\quad & x = \pm\sqrt{-20}\text{, which are not real numbers.} \end{aligned}$$



What do we conclude here? Well, we tried to figure out what values of x would make the denominator zero, so that we could exclude such an x from being in the domain. But there aren't any values of x that make the denominator zero. So there is nothing to exclude from the domain. Therefore, every real number is allowed in the function. The domain is

$$\boxed{\mathbb{R}}$$

Homework

Find the domain of each function:

13. $y = \pi$

14. $y = \dfrac{x^2 - 9}{9x - 7}$

15. $y = \dfrac{2x + 1}{x^2 - 100}$

16. $y = \dfrac{x^2 - 25}{x^2 + 49}$

17. $y = \sqrt{2} + \sqrt{3}$

18. $y = \sqrt{3}$

19. $y = \dfrac{2x - 3}{2x^2 + 3x}$

20. $y = \dfrac{3}{2x - x^2}$

21. $y = x^3 - 8$

22. $y = \dfrac{9x - 7}{2x + 9}$

23. $y = \dfrac{5x + 25}{x^2 - 144}$

24. $y = \dfrac{x}{x^2 + 3}$

25. $y = \sqrt{2}\,x + 9$

26. $y = \dfrac{x - 2}{x^2 - 81}$

27. $y = \dfrac{3}{2x^2 - 5x - 3}$

28. $y = 9$

29. $y = \dfrac{1}{x^2 + 10x + 25}$

30. $y = \dfrac{2x - 7}{8x^2 - 10x - 3}$

31. $y = \dfrac{x + 10}{x^2 + 100}$

32. $y = \dfrac{x^2 + 5x + 1}{4x^2 - x - 3}$

33. $y = \dfrac{3x + 7}{10x^2 - 20x}$

34. $y = \dfrac{x^2 - 14}{14x^2 - 42x}$

❑ THE DOMAIN OF SQUARE ROOT FUNCTIONS

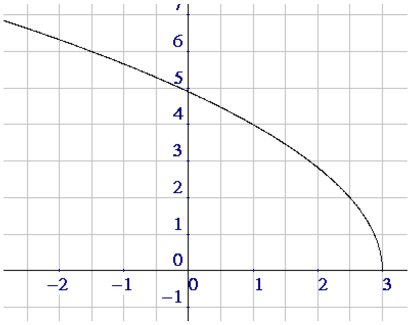
EXAMPLE 1: **Find the domain of $y = \sqrt{24-8x}$.**

Solution: In this formula, we ask ourselves if anything could possibly go wrong using certain values of x. Well, the square root of a negative number doesn't exist in the real numbers, so certainly something could go wrong — for instance, if $x = 4$, then we'd have $y = \sqrt{-8}$.

Now we need a statement that describes the legal x's. How about this:

The square root of a quantity is defined (as a real number) only when that quantity is greater than or equal to zero.

So, in this example, the radicand, $24 - 8x$, must be greater than or equal to 0; i.e., **≥ 0**



$$24 - 8x \geq 0 \qquad \textbf{(the radicand must be 0 or positive)}$$

$$\Rightarrow \quad -8x \geq -24 \qquad \text{(subtract 24 from each side)}$$

$$\Rightarrow \quad x \leq 3 \qquad \text{(divide by } -8\text{, and reverse the inequality)}$$

Therefore, the domain is the set of all real numbers which are less than or equal to 3:

$$\boxed{x \leq 3}$$ or, **(–∞, 3]** in interval notation

Homework

Find the domain of each function:

35. $y = \sqrt{3x+12}$

36. $y = \sqrt{8-2x}$

37. $y = \sqrt{7x-15}$

38. $y = \sqrt{-4x+1}$

39. $y = \sqrt{17-17x}$

40. $y = \sqrt{4x+12}$

41. $y = \sqrt{4x+1}$

42. $y = \sqrt{10-10x}$

43. $y = \sqrt{20x}$

44. $y = \sqrt{20-x}$

45. $y = \sqrt{-3x-15}$

46. $y = \sqrt{-30-7x}$

Review Problems

Find the domain of each function:

47. $y = \sqrt{2}$

48. $y = \dfrac{2x-6}{x^2+4x+3}$

49. $f(x) = \pi$

50. $y = \pi x^3 - 17x + 1$

51. $y = \dfrac{x+2}{x-3}$

52. $y = \dfrac{3}{x^2-x-20}$

53. $y = \dfrac{20}{x^2 - x}$

54. $y = \dfrac{\pi}{13x^2 + 26x}$

55. $y = \sqrt{18 - 9x}$

56. $y = \dfrac{4 - x}{x^2 - 400}$

57. $y = \dfrac{13x - 13}{9x^2 - 42x + 49}$

58. $y = \dfrac{17x - 17}{25x^2 + 20x + 4}$

59. $y = \dfrac{x^2 - 5x + 6}{50}$

60. $y = \dfrac{\sqrt{2}}{3}$

61. $y = \sqrt{12x - 16}$

62. $y = \sqrt{10 - 5x}$

63. $y = \dfrac{x^2 - 49}{2x + 17}$

64. $y = \dfrac{\sqrt{3}}{9x^2 - 49}$

Solutions

1. a. 33/2 b. 105/4 c. 21 d. Undefined
e. Undefined <u>Domain</u>

2. [3, 5]

3. ℝ

4. All four numbers are legal inputs, since all four numbers could be multiplied by 2 and then have 10 added on with no problem.

5. 3 is a legal input, since $\frac{3}{3-2} = \frac{3}{1} = 3$; no muss, no fuss.
2 is not legal, since $\frac{3}{2-2} = \frac{3}{0} = \textit{Undefined}$.
π is legal because nothing can go wrong.
0 is legal because $\frac{3}{-2}$ is a perfectly fine number.

6. $\sqrt{7}$ is a perfectly fine real number, so 7 is a legal input.
$\sqrt{0}$ is also a real number (= 0), so 0 is legal.
$\sqrt{-9}$ is not a real number, so this number doesn't exist in our current world; hence, –9 is not a legal input.
$\sqrt{99}$ is some real number (9 point something), so 99 is legal.

7. When $x = 12$, we get $12^2 - 144 = 144 - 144 = 0$ in the denominator. Zero in the denominator? I don't think so! Thus, 12 is not a legal input. 20 is O.K. to use for x, just as 0 is fine for x. So 20 and 0 are legal inputs. But when $x = -12$, we're in the same boat as when x was 12. This is because when $x = -12$ we get $144 - 144 = 0$ in the denominator, which is certainly not allowed. Therefore, –12 is not legal.

8. All four values of x are legal, since none of them makes the denominator zero.

9. a. 4 b. 8 c. 1 d. 0 e. Undefined

10. 1/7; Undefined; –1/5; –1/9; –1/8; Undefined; 1/7; 1/16
When $x = 3$ or -3, the function is undefined due to dividing by zero. It seems that no other inputs would produce a zero in the denominator. So, our guess would be that the domain of h is all real numbers except 3 and –3. This domain can be written $\mathbb{R} - \{3, -3\}$, or $\mathbb{R} - \{\pm 3\}$

11. $\mathbb{R}$ **12**. $\mathbb{R} - \{-3\}$ **13**. $\mathbb{R}$ **14**. $\mathbb{R} - \left\{\frac{7}{9}\right\}$ **15**. $\mathbb{R} - \{\pm 10\}$

16. $\mathbb{R}$, and here's why: No matter what x is, x^2 is at least 0 (since x^2 is never negative). Now add 49 to something that is at least 0, and you now have a number which is at least 49; so the denominator $x^2 + 49$ can never be 0. Since dividing by 0 is the only critical issue in this function, the domain is all real numbers.

17. $\mathbb{R}$
18. $\mathbb{R}$
19. $\mathbb{R} - \left\{0, -\frac{3}{2}\right\}$
20. $\mathbb{R} - \{0, 2\}$
21. $\mathbb{R}$
22. $\mathbb{R} - \left\{-\frac{9}{2}\right\}$
23. $\mathbb{R} - \{\pm 12\}$
24. $\mathbb{R}$
25. $\mathbb{R}$
26. $\mathbb{R} - \{\pm 9\}$
27. $\mathbb{R} - \left\{3, -\frac{1}{2}\right\}$
28. $\mathbb{R}$
29. $\mathbb{R} - \{-5\}$
30. $\mathbb{R} - \left\{-\frac{1}{4}, \frac{3}{2}\right\}$
31. $\mathbb{R}$
32. $\mathbb{R} - \left\{1, -\frac{3}{4}\right\}$
33. $\mathbb{R} - \{0, 2\}$
34. $\mathbb{R} - \{0, 3\}$
35. $x \ge -4$
36. $x \le 4$
37. $x \ge \frac{15}{7}$
38. $x \le \frac{1}{4}$
39. $x \le 1$
40. $x \ge -3$
41. $x \ge -\frac{1}{4}$
42. $x \le 1$
43. $x \ge 0$
44. $x \le 20$
45. $x \le -5$
46. $x \le -\frac{30}{7}$
47. $\mathbb{R}$
48. $\mathbb{R} - \{-1, -3\}$
49. $\mathbb{R}$
50. $\mathbb{R}$
51. $\mathbb{R} - \{3\}$
52. $\mathbb{R} - \{5, -4\}$
53. $\mathbb{R} - \{0, 1\}$
54. $\mathbb{R} - \{0, -2\}$
55. $x \le 2$
56. $\mathbb{R} - \{\pm 20\}$
57. $\mathbb{R} - \left\{\frac{7}{3}\right\}$
58. $\mathbb{R} - \left\{-\frac{2}{5}\right\}$
59. $\mathbb{R}$
60. $\mathbb{R}$
61. $x \ge \frac{4}{3}$
62. $x \le 2$
63. $\mathbb{R} - \left\{-\frac{17}{2}\right\}$
64. $\mathbb{R} - \left\{\pm\frac{7}{3}\right\}$

<u>NOTE:</u> All of the domains in the solutions involving inequalities could also be written in Interval Notation. Examples:

$x \ge 4$ can be written $[4, \infty)$.

$x < -3$ can be written $(-\infty, -3)$.

CH N –THE NUMBER *e*

Recall the special number π. It is the ratio of the circumference of any circle to its diameter, and we learned that it's irrational (an infinite, non-repeating decimal). There's another extremely important irrational number and it's the topic of this chapter. It will be defined in terms of a limit, and will be the basis of applications in banking, biology, and radioactivity. In Calculus, it's absolutely the most important base for exponential functions.

☐ *NEW LIMIT NOTATION*

Let's review an example of limits. Consider the function

$$f(x) = \frac{1}{x}.$$

We know that as x grows larger and larger, $f(x)$ gets closer and closer to 0. We usually write:

$$\text{As } x \to \infty,\ f(x) \to 0.$$

In our new notation, we would write this fact as

$$\lim_{x \to \infty} f(x) = 0.$$

In general, instead of writing

$$\text{As } x \to a, f(x) \to L,$$

we write

$$\lim_{x \to a} f(x) = L.$$

These two statements mean exactly the same thing.

Homework

1. If $h(x) = x^2$, find $\lim\limits_{x \to 3} h(x)$.

2. If $f(x) = \dfrac{2x+1}{x-3}$, find $\lim\limits_{x \to \infty} f(x)$. Hint: Consider the graph of f.

3. If $E(x) = \dfrac{9}{1+x^2}$, find $\lim\limits_{x \to 0} E(x)$.

☐ *EARNING INTEREST AT THE BANK*

In **simple interest** at the bank, the interest you earn on your investment is calculated just once at the end of the year. Suppose you place \$70 in the bank at 10% interest per year. At the end of the year you will have earned \$7 in interest, making your current balance \$77.

Now let's consider **compound interest**. As an example, we'll divide the year into four compounding periods (each one called a *quarter*), and we'll pretend we have \$1 invested at 100% annual interest. At the end of each quarter, you earn interest in the amount of 25% (100% divided into 4 equal parts) of your current balance. This means that from now on, your interest itself is earning interest.

Start: **\$1**

1st Quarter: previous balance + 25% of the previous balance

$= \$1 + 25\% \text{ of } \$1$

$= \$1 + \$\frac{1}{4}$ (**balance = \$1.25** after 1 quarter)

2nd Quarter: previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4}) + 25\% \text{ of } (\$1 + \$\frac{1}{4})$

$= (\$1 + \$\tfrac{1}{4}) + \tfrac{1}{4}(\$1 + \$\tfrac{1}{4})$

$= (\$1 + \$\tfrac{1}{4})(\$1 + \$\tfrac{1}{4})$ (factor out $1 + \tfrac{1}{4}$)

$= (\$1+\$\tfrac{1}{4})^2$ (**balance = $1.56** after 2 quarters)

3rd Quarter: previous balance + 25% of the previous balance

$= (\$1+\$\tfrac{1}{4})^2 + 25\%$ of $(\$1+\$\tfrac{1}{4})^2$

$= (\$1+\$\tfrac{1}{4})^2 + \tfrac{1}{4}(\$1+\$\tfrac{1}{4})^2$

$= (\$1+\$\tfrac{1}{4})^2 (\$1+\$\tfrac{1}{4})$ (factor out $(1 + \tfrac{1}{4})^2$)

$= (\$1+\$\tfrac{1}{4})^3$ (**balance = $1.95** after 3 quarters)

4th Quarter: previous balance + 25% of the previous balance

$= (\$1+\$\tfrac{1}{4})^3 + 25\%$ of $(\$1+\$\tfrac{1}{4})^3$

$= (\$1+\$\tfrac{1}{4})^3 + \tfrac{1}{4}(\$1+\$\tfrac{1}{4})^3$

1 $= (\$1+\$\tfrac{1}{4})^3 (\$1+\$\tfrac{1}{4})$ (factor out $(1 + \tfrac{1}{4})^3$)

$= (\$1+\$\tfrac{1}{4})^4$ (**balance = $2.44** after 1 full year)

Note: To compare simple interest with compound interest, $1.00 invested for one year at 100% simple interest would yield $1.00 in interest, for a final balance of **$2.00**. Using compound interest, we see that the last calculation in the 4th quarter above is

$$(\$1+\$\tfrac{1}{4})^4 = \$1.25^4 = \mathbf{\$2.44}$$

To summarize, investing $1 at 100% annual interest compounded four times a year yields a final balance, after one year in the bank, of

$$(\$1+\$\tfrac{1}{4})^4$$

From this example we can generalize: Investing $1 at 100% annual interest compounded ***n*** times a year yields a final balance of

$$\left(\$1 + \$\frac{1}{\boldsymbol{n}}\right)^{\boldsymbol{n}}$$

at the end of one year. For example, investing \$1 at 100% annual interest compounded 360 times a year would give you a balance of

$$\left(1 + \frac{1}{\mathbf{360}}\right)^{\mathbf{360}} = (1.002777778)^{360} = \mathbf{\$2.7145}$$

Summarizing,

Investing \$1 at 100% annual interest for one year, compounded *n* times a year, yields a final balance of

$$\left(1 + \frac{1}{n}\right)^{n}$$

Homework

4. For each problem, use the formula $B = \left(1 + \frac{1}{n}\right)^{n}$ to find the balance at the end of one year if the original investment was \$1, earning 100% annual interest, and assuming the given number of compounding periods:

a. 1 b. 12 c. 365 d. 1,000 e. 5,000,000

☐ THE MOST IMPORTANT EXPONENTIAL BASE OF ALL !

Now suppose that we consider compounding so often that it occurs at every instant of time. This is called ***continuous compounding***. What will your account balance be after one year of continuous compounding? It would come from the formula

$$\left(1+\frac{1}{n}\right)^n$$

with n replaced by ∞: $\left(1+\frac{1}{\infty}\right)^{\infty}$. But this would be meaningless, so we use the new limit notation introduced a few pages back. We keep the n's in the formula, and then specify that n should approach infinity; that is, n becomes infinitely large:

Whatever every instant of time means, it drove the Greeks bonkers, and it wasn't until the 17th century that Newton and Leibniz made some sense of this when they invented Calculus.]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It's hard to see this now, but this limit is actually a specific real number. In fact, this number is one of the most important numbers in math, science, statistics, and business. It is given the name "***e***", perhaps due to the word "***e***xponential," or perhaps due to its creator Leonard ***E***uler.

Though ***e*** is defined as a limit, a decimal approximation of ***e*** will be discussed in the homework. In summary,

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

Homework

5. Define e in two ways: as a limit, and as the result of a certain investment.

6. The number e is irrational, although I have no idea how to prove it. Nevertheless, explain exactly what it means for e to be irrational.

7. a. Use the definition of e with a value of $n = 10{,}000{,}000$ to approximate the value of e.

   b. Use the e^x button on your calculator to approximate e. Hint: $e = e^1$.

   c. Which approximation do you think is more accurate?

8. Use your calculator to approximate each of the following:

   a. $e^{3.7}$ b. $\dfrac{1}{e}$ c. $\sqrt[7]{e}$ d. $\dfrac{e^2 - \sqrt{e}}{\sqrt[3]{4e+1}}$

9. Some books define ***e*** as $\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^n$. Prove that this is the same as our definition.

☐ *THE LAWS OF EXPONENTS*

All the laws for exponents with which we're familiar work for exponential expressions like 2^x and e^x just as well as they did for x^2. Here are the six main rules. Assume a and b are positive constants.

$$a^{-x} = \frac{1}{a^x} \qquad a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Homework

10. Simplify each expression:

a. $2^x 2^y$ b. e^0 c. e^{-5} d. $\frac{e^5}{e^2}$ e. $\frac{e^x}{e^y}$

f. $(e^x)^e$ g. $(10e)^x$ h. $\left(\frac{e}{2}\right)^n$ i. $(e^{\pi})^{10x}$ j. $2^3 2^0$

Review Problems

11. By making a sketch of the function $y = \frac{2}{x}$, calculate:

a. $\lim\limits_{x \to \infty} y$ b. $\lim\limits_{x \to -\infty} y$

12. a. Define *e* as it applies to continuous compounding in the bank.
b. Define *e* as a limit.
c. Is *e* a real number?
d. Is *e* rational or irrational?
e. Explain exactly why *e* is a valid base for an exponential function.
f. Is $e > \pi$ or is $e < \pi$?

13. a. Simplify: $e^0 + \frac{e^7}{e^2} + (e^3)^4$ b. Approximate: $e^3 + \sqrt[3]{e}$
c. Approximate: $(\pi - e)^2$ d. Approximate: e^{π}.

Solutions

1. In this limit, x is approaching 3. This means that x is approaching 3 by taking values like 3.2, 3.1, 3.05, 3.001, 3.000001 – or perhaps x is taking values like 2.9, 2.95. 2.99, 2.999, etc. Now, as x does this, what does the functional value x^2 do? Well, it gets closer and closer to 9. Therefore, the limit is 9.

2. Since x is approaching infinity, this limit is equivalent to finding a horizontal asymptote. Take your calculator and evaluate the function for a really huge value of x. You should see that the limit is 2.

3. As x shrinks toward 0, the x^2 term in the denominator essentially disappears, leaving a functional value of 9, which is the answer to the limit question.

4. a. \$2.00 b. \$2.6130 c. \$2.7146 d. \$2.7169 e. \$2.7183

5. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

If you invest \$1 at a 100% interest rate compounded continuously, your balance at the end of one year will be \$$e$.

6. e is an infinite, non-repeating decimal.

7. a. My TI-30X gives the answer 2.718288827, but your calculator might give 2.718281693, or something close.

b. 2.718281828 c. The second value

8. a. 40.4473 b. 0.3679 c. 1.1536 d. 2.5162

9. Since $1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$, the definitions are equivalent.

10. a. 2^{x+y} b. 1 c. $\frac{1}{e^5}$ d. e^3 e. e^{x-y}

f. e^{ex} g. $10^x e^x$ h. $\frac{e^n}{2^n}$ i. $e^{10\pi x}$ j. 8

11. a. 0 b. 0

12. a. Invest $1 for 1 year at an interest rate of 100%/yr compounded <u>continuously</u>. At the end of the year, the account balance is $*e*.

b. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ c. Yes d. Irrational

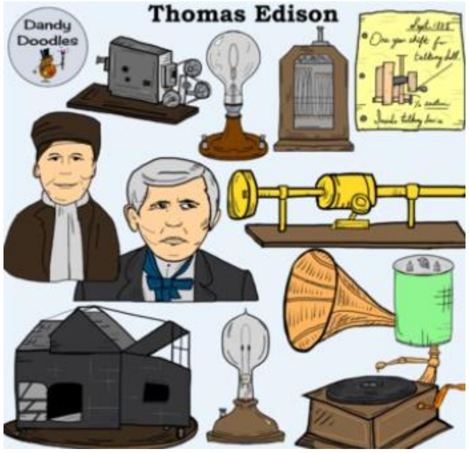
e. *e* is real number that is greater than 0 but not equal to 1.

f. $e < \pi$

13. a. $1 + e^5 + e^{12}$ b. 21.4811 c. 0.1792 d. 23.1407

10

> "Many of life's failures are people
> who did not realize
> how close they were to success
> when they gave up."
>
> – *Thomas Edison*



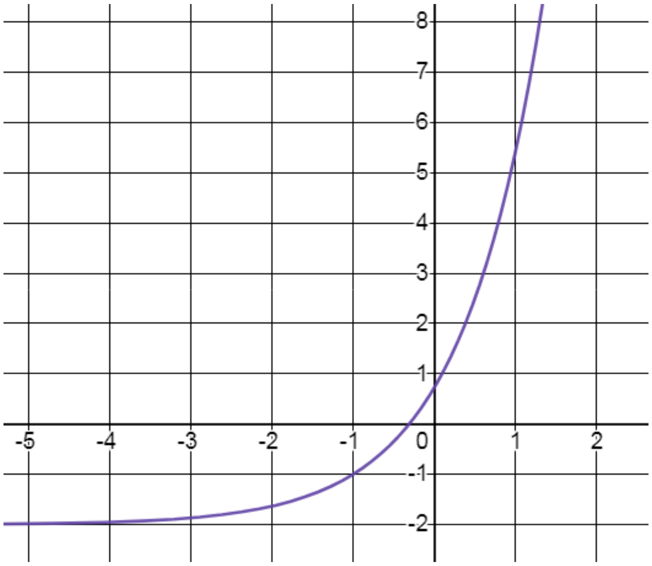
CH NN – GRAPHING WITH BASE *e*

Since *e* is a positive real number not equal to 1 (it's greater than 1), it's allowed to be the base of an exponential function. In fact, the base-*e* exponential function, $y = e^x$, is one of the most important exponential functions in business, biology, chemistry, ecology, statistics, physics, and electronics. Let's graph one of these.

EXAMPLE 1: **Graph:** $y = e^{x+1} - 2$

Solution: Use your calculator to verify that each of the following ordered pairs is on the graph of the function:

(–4, –1.95) (–3, –1.86) (–2, –1.63) (–1, –1) (0, 0.72) (1, 5.39)



Exponential Growth

The **domain** of the function is $\mathbb{R}$. To determine the range, we need to verify what the graph appears to show, that the line $y = -2$ is a horizontal asymptote. Let x be a large negative number, for instance $x = -20$. Then the y-value is

$$y = e^{-20+1} - 2 = e^{-19} - 2 = .000000006 - 2 = -\mathbf{1.999999994}$$

which is extremely close to (and slightly above) –2. We can be pretty confident that the **horizontal asymptote** is $\boldsymbol{y = -2}$. From this fact, we can conclude that the **range** of the function is all real numbers greater than –2: **(–2, ∞)**.

For our last bit of analysis, we summarize the behavior of the function at extreme values of x by writing the limits

$$\lim_{x \to \infty} (e^{x+1} - 2) = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} (e^{x+1} - 2) = -\mathbf{2}$$

The first of these two limits indicates that the graph is **increasing**; that is, as you move from left to right along the graph, the graph grows taller and taller (and never levels off). We can call this **exponential growth**. In our following example, we change one little (but critical) part of the equation (which part?) and we get a curve which represents **exponential decay**.

EXAMPLE 2: **Graph:** $\boldsymbol{y = e^{-x} + 1}$

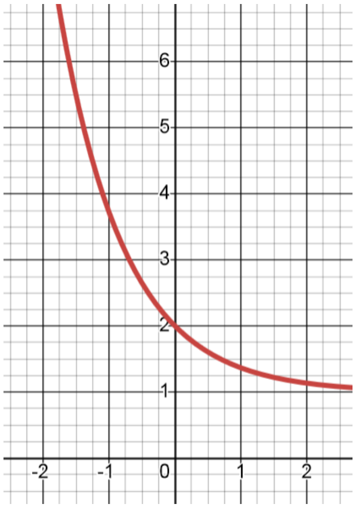
Solution: Let's go for the y-intercept first. Letting $x = 0$ produces $y = e^{-0} + 1 = e^0 + 1 = 1 + 1 = 2$. The y-intercept is therefore **(0, 2)**. Use your calculator to verify some other points on the graph:

(–3, 21.09) (–2, 8.39) (–1, 3.72) (1, 1.37) (2, 1.14) (3, 1.05)

To emphasize the limits as x approaches ∞ or –∞, let's calculate two more points:

(–10, 22027.47) (10, 1.000045)

It appears that as $x \to -\infty$, $y \to \infty$. Also, as $x \to \infty$, $y \to 1$.



Exponential Decay

The domain of the function is $\mathbb{R}$, simply because any real number x can be used in the formula without a problem.

The range is all real numbers greater than 1: **$(1, \infty)$**

From the graph and the points we calculated, we see that there's a horizontal asymptote at **$y = 1$**.

Homework

Graph each of these exponential functions:

1. $y = e^x$
2. $f(x) = e^{-x}$
3. $g(x) = e^2 + 2$
4. $h(x) = e^{x-2}$
5. $f(x) = e^x - 1$
6. $y = e^{x-1} + 2$
7. $E(x) = e^{x+2} - 3$
8. $y = e^{-2x}$
9. $y = e^{-x} - 1$
10. $h(x) = e^{-x} + 2$
11. Describe the graph of $y = ex + e$.

❑ THE BELL-SHAPED CURVE

Some of you will take Statistics soon, and you will learn that the most important function in the course is a special bell-shaped curve called the ***standard normal*** curve (or Gaussian Curve). It's an exponential function, and its formula is a bit complicated, but we have all the tools needed to find some of its points and then sketch it.

The formula for the standard normal curve is given by

$$\boxed{y = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}}$$

Note that this function is a type of ***exponential function*** because the base is a constant (in this case, *e*) and the variable *x* occurs in the exponent. Let's decipher this formula before we try to plot any points: The coefficient of the exponential function is $\frac{1}{\sqrt{2\pi}}$, the base is *e*, and the exponent on the *e* is $-\frac{1}{2}x^2$.

Let's calculate some (x, y) pairs for this function and see what kind of graph we get. Let's start by letting $x = 0$. This, of course, will yield the *y*-intercept:

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}e^{0} = \frac{1}{\sqrt{2\pi}}(1) \approx 0.3989$$

This gives us the approximate point **(0, 0.4)**. We're off to a good start.

Now we'll choose $x = 1$. Plugging this number into the function gives

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(0.6065) \approx 0.2420$$

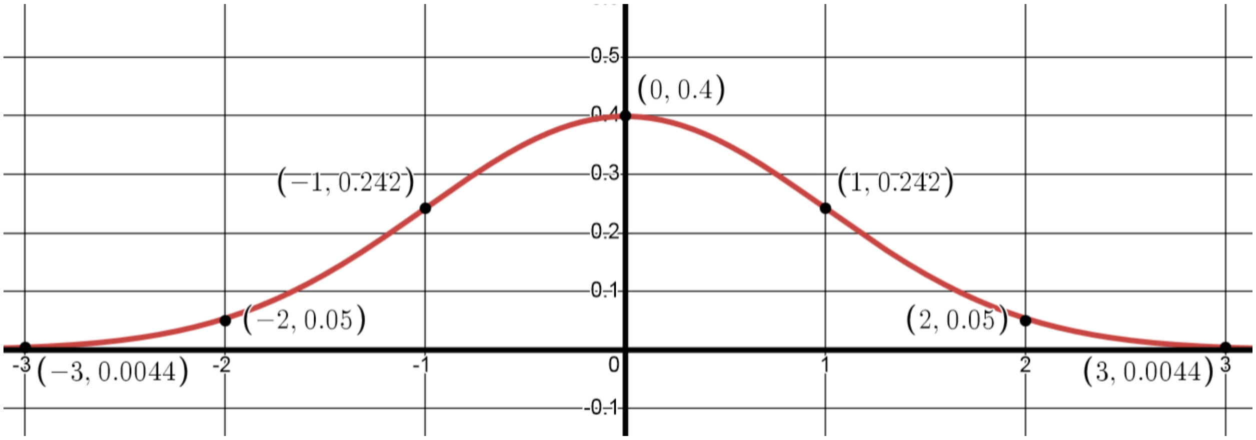
Our second point is therefore **(1, 0.24)**. Now for $x = -1$. You should be able to do this one in your head. First, the –1 is squared, giving 1 — but this is exactly what we had in the previous calculation, and so the *y*-value must also be 0.2420. Our third point is therefore **(–1, 0.24)**.

Choosing $x = 2$ (or –2) gives the following calculation for *y*:

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\pm 2)^2} = \frac{1}{\sqrt{2\pi}}e^{-2} = \frac{1}{\sqrt{2\pi}}(0.1353) \approx 0.0540$$

We now have two more points for our graph: **(2, 0.05)** and **(–2, 0.05)**. Continuing in this manner, we can find the points **(3, 0.0044)** and

(–3, 0.0044). It appears that as we get farther from the origin, the y-values are getting smaller and smaller, but always remaining positive. If we plot the seven points we've just computed and then connect them with a smooth curve, we get the following:



What can we infer from this graph? Lots of stuff:

A. The **domain** is ℝ, all real numbers. [The graph goes infinitely to the right and infinitely to the left.] This means that x can take on any value at all and the formula will work out just fine.

B. The **range** is all numbers that are bigger than 0 but less than or equal to *approximately* 0.4. In interval notation: **(0, 0.4]**.

C. If we fold the portion of the graph in Quadrant I over into Quadrant II, the curves will coincide. We therefore have ***y*-axis symmetry**. [You may have learned to change x to $-x$ and see if you get the same formula; you do.]

D. The calculations and the graph indicate these two limits:

$$\lim_{x \to \pm\infty} y = \mathbf{0}$$

E. And from the limits we deduce that the line $\boldsymbol{y = 0}$ (the x-axis) is a **horizontal asymptote** of the graph.

F. One reason for some of the numbers in the function is that we want the total **area** between the curve and the x-axis to be **1**.

Review Problems

12. a. List the domain, range, intercepts, and asymptotes of $y = e^x$.

    b. $\lim\limits_{x\to\infty} e^x$ c. $\lim\limits_{x\to-\infty} e^x$ d. $\lim\limits_{x\to 0} e^x$

13. Let $f(x) = e^x$ and $g(x) = e^{x+\pi} + 100$. Describe the graph of g relative to the graph of f.

14. Graph: $y = e^{x-2} - 3$. Calculate the y-intercept and any asymptotes.

15. Graph: $y = e^{-x} - 3$. Calculate the y-intercept and any asymptotes.

16. Graph: $N(x) = e^{-x^2}$. Hint: If $x = 1$, then $e^{-1^2} = e^{-1} = \frac{1}{e} \approx 0.3679$. Prove that the graph has y-axis symmetry.

17. Explain why the graph of $y = x^2 + ex + \pi$ is a parabola, and then prove that it has no x-intercepts.

18. True/False:

a. $\lim\limits_{x\to\infty} f(x) = L$ means the same thing as "As $x \to \infty$, $f(x) \to L$."

b. If $P(x) = \dfrac{6}{1+x^2}$, then $\lim\limits_{x\to 0} P(x) = 6$.

c. The function $y = e^x$ is a polynomial.

d. The domain of the above function is $\mathbb{R}$.

e. The range of the above function is $(0, \infty)$.

f. Compared to the graph of $y = e^x$, the graph of $y = e^x + 2$ is two units higher.

g. The function $y = e^{-x}$ is decreasing.

h. The function $y = e^{-x}$ has a horizontal asymptote.

i. The function $y = e^{-x}$ has a vertical asymptote.

j. The slope of the line $y = \pi x + e$ is e.

k. The y-intercept of the graph of

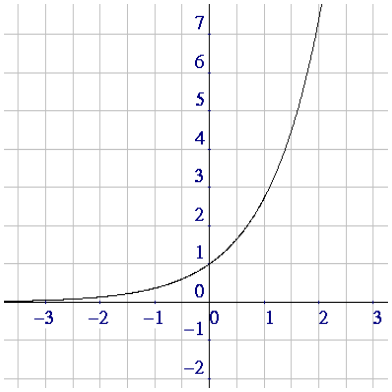
$$h(x) = e^{1-x} + 2$$

is $(0, e + 2)$.

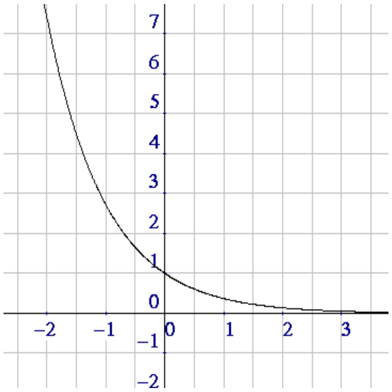
l. $e > \pi$

Solutions

1.

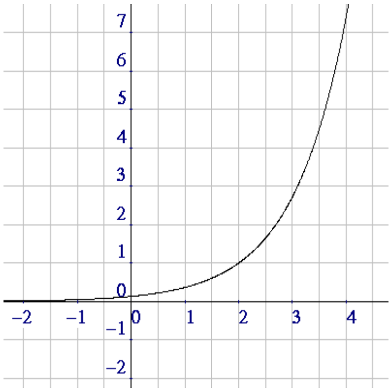


2.

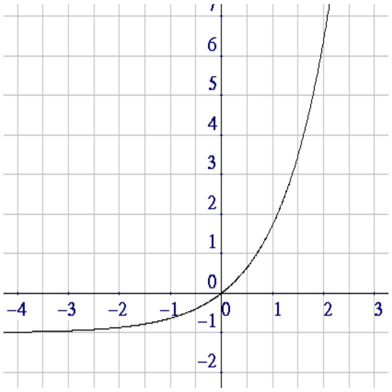


3. I'd like to see what you think the graph is.

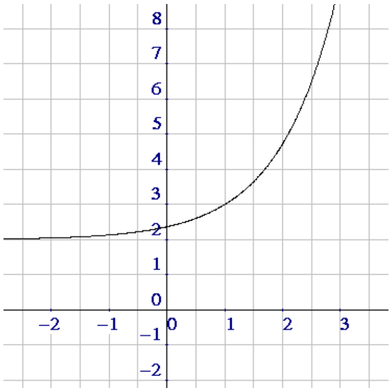
4.



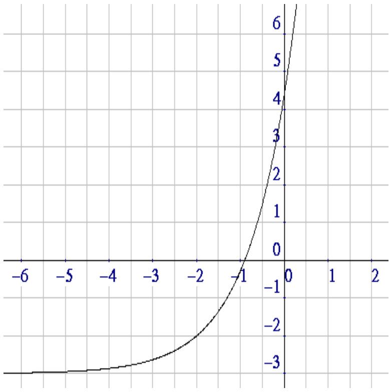
5.



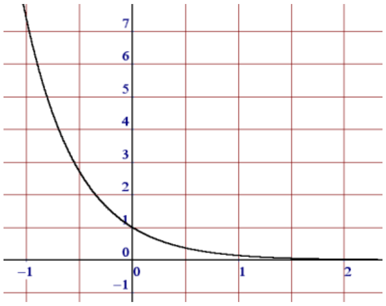
6.



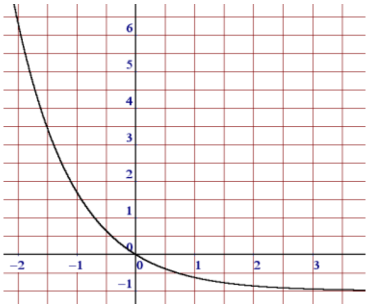
7.



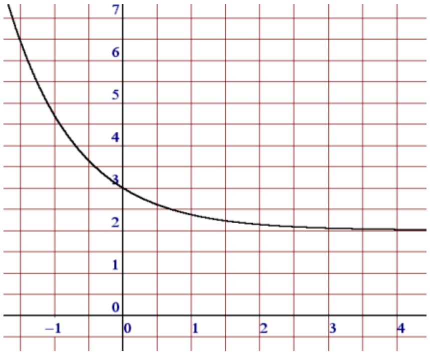
8.



9.



10.

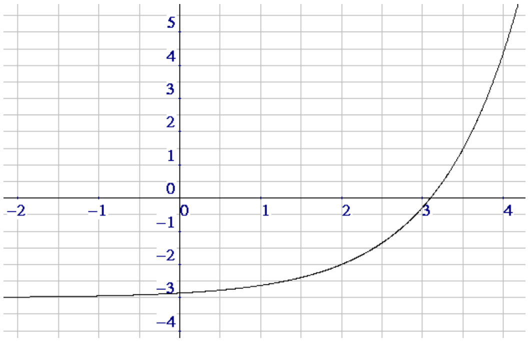


11. The graph of $y = ex + e$ is a straight line with slope e and having a y-intercept of $(0, e)$.

12. a. Domain = $\mathbb{R}$; Range = $(0, \infty)$;
x-int: none; y-int: $(0, 1)$;
vert asy: none; horiz asy: $y = 0$
b. ∞ c. 0 d. 1

13. The graph of g is the graph of f shifted π units to the left and 100 units up.

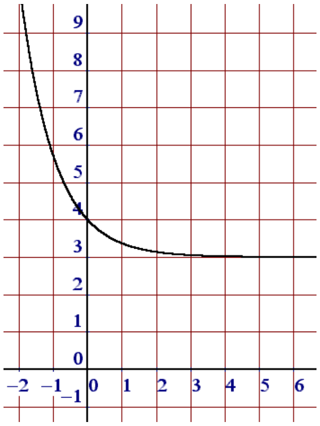
14.



y-int: $(0, e^{-2} - 3) \approx (0, -2.86)$

horiz asy: $y = -3$

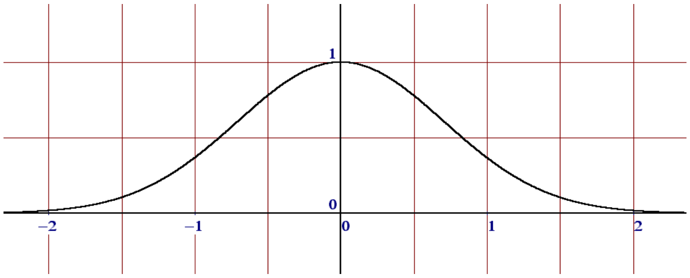
15.



y-int: (0, 4)

horiz asy: $y = 3$

16.



To prove y-axis symmetry, we change each x in the formula to $-x$ and see what we get: $e^{-(-x)^2} = e^{-x^2}$, the same formula. We're done.

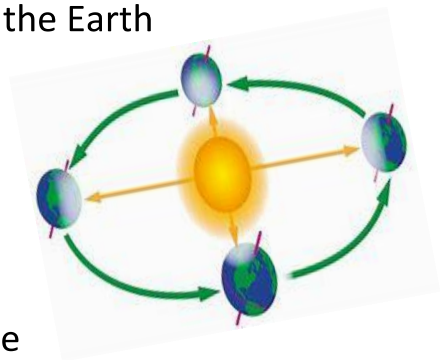
17. It's a parabola because it fits the form $y = ax^2 + bx + c$; in fact, $a = 1$, $b = e$, and $c = \pi$.
To find x-intercepts, set y to 0; solve the resulting quadratic equation for x using the Quadratic Formula. You should get a negative radicand, indicating no solutions in $\mathbb{R}$, and thus no x-intercepts.

18. a. T b. T c. F d. T e. T f. T g. T h. T
i. F j. F k. T l. F

"The direction in which education starts a man will determine his future life." Plato

CH XX –THE ELLIPSE

An ***ellipse*** is oval-shaped and may be described as a circle that never quite made it. It's the shape of orbits. For example, the orbit of the Earth around the Sun is an ellipse, and the orbits of many satellites (including our Moon) revolving around the Earth are elliptical. The elliptical shape is also used in certain gears, and it is the key behind a medical procedure called *lithotripsy* (Greek for stone-crushing) — the use of shock waves to break up kidney stones.



☐ *FINDING INTERCEPTS*

In this section, we recall the procedures for finding intercepts; the practice will help us with the following ellipse problems.

EXAMPLE 1: **Find all the intercepts of the graph**

$$\mathbf{49x^2 + 9y^2 = 441}$$

Solution: To find any x-intercepts, we set $y = 0$ and solve for x:

$$49x^2 + 9y^2 = 441 \Rightarrow 49x^2 + 9(\mathbf{0})^2 = 441$$

$$\Rightarrow 49x^2 = 441 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

Thus, the x-intercepts are **(3, 0)** and **(–3, 0)**.

To find any y-intercepts, we set $x = 0$ and solve for y:

$$49x^2 + 9y^2 = 441 \;\Rightarrow\; 49(\mathbf{0})^2 + 9y^2 = 441$$

$$\Rightarrow\; 9y^2 = 441 \;\Rightarrow\; y^2 = 49 \;\Rightarrow\; y = \pm 7$$

Therefore, the y-intercepts are **(0, 7)** and **(0, –7)**.

Homework

1. Find all the **intercepts** of each graph:

a. $3x + 5y = 30$

b. $y = 4x + 20$

c. $x = 3$

d. $y = -4$

e. $y = x^2$

f. $y = x^2 - x - 6$

g. $x^2 + y^2 = 121$

h. $x^2 + y^2 = 50$

i. $9x^2 + y^2 = 9$

j. $x^2 + 36y^2 = 36$

k. $81x^2 + 25y^2 = 2025$

l. $x^2 + 4y^2 = 16$

m. $x^2 - y^2 = 100$

n. $y^2 - x^2 = 121$

o. $y = x^2 + x + 1$

❑ *GRAPHING ELLIPSES*

EXAMPLE 2: **Graph the ellipse:** $\boldsymbol{\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1}$

Solution: Let's begin by finding all the intercepts of the ellipse. These points, and a few others, will convince us that the graph is indeed elliptical.

***x*-intercepts:** Set $y = 0$ to get

$$\frac{x^2}{4} + \frac{\mathbf{0}^2}{25} = 1 \;\Rightarrow\; \frac{x^2}{4} = 1 \;\Rightarrow\; x^2 = 4 \;\Rightarrow\; x = \pm\sqrt{4} = \pm 2$$

and so the x-intercepts are **(2, 0)** and **(–2, 0)**.

***y*-intercepts:** Set $x = 0$ to get

$$\frac{\mathbf{0}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = 1 \;\Rightarrow\; y^2 = 25 \;\Rightarrow\; y = \pm\sqrt{25} = \pm 5$$

and thus the y-intercepts are **(0, 5)** and **(0, –5)**.

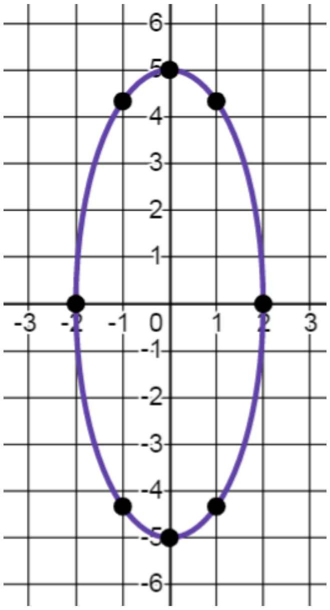
Additional points: Let $x = 1$. Then

$$\frac{\mathbf{1}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{1}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = \frac{3}{4} \;\Rightarrow\; y^2 = \frac{75}{4}$$

$$\Rightarrow\; y = \pm\sqrt{\frac{75}{4}} \;\Rightarrow\; y = \pm\sqrt{18.75} \approx \pm 4.33$$

This yields the points **(1, 4.33)** and **(1, –4.33)**.

Note that if we let $x = -1$, we obtain the same y-values. Thus, two additional points are **(–1, 4.33)** and **(–1, –4.33)**. [Can you explain why?] We now have a total of eight points we can plot. When we do, we get the graph of the ellipse:



Note: The **center** of the ellipse

$$\boldsymbol{\frac{x^2}{4} + \frac{y^2}{25} = 1}$$

is the **origin**.

A Final Note:

The graph shows that an x-value of 3, for example, should not be allowed in the formula (since there's no point on the ellipse whose x-coordinate is 3). But how can we verify this fact before we graph anything? We'll let $x = 3$ in the ellipse equation and see what happens:

$$\frac{x^2}{4}+\frac{y^2}{25} = 1 \quad \overset{x=3}{\Rightarrow} \quad \frac{\mathbf{3}^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{9}{4}+\frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{y^2}{25} = 1-\frac{9}{4} \quad \Rightarrow \quad \frac{y^2}{25} = -\frac{5}{4} \quad \Rightarrow \quad y^2 = -\frac{125}{4}$$

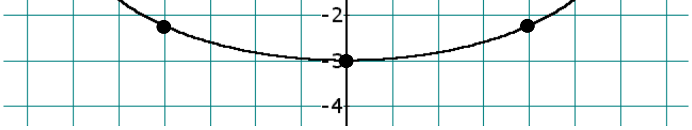
$$\Rightarrow \quad y = \pm\sqrt{-\frac{125}{4}}$$, which are <u>not</u> real numbers.

Since letting $x = 3$ resulted in no real-number solution for y, we conclude that no point on the graph will have an x-coordinate of 3, and thus 3 is not an allowable x-value; it's not in the ellipse's domain.

<u>EXAMPLE 3:</u> **Graph the ellipse:** $\boldsymbol{\frac{x^2}{36}+\frac{y^2}{9} = 1}$

<u>Solution:</u> Since this equation is very similar to the previous one, its graph is likely an ellipse with center at the origin. You can calculate the intercepts to be **(6, 0)**, **(–6, 0)**, **(0, 3)**, and **(0, –3)**.

For extra "resolution," we can find four more points by letting $x = \pm4$, and then calculating y to be $\pm\sqrt{5}$. We thus get four points with the approximate values (**4, 2.24**), (**4, –2.24**), (**–4, 2.24**), and **(–4, –2.24)**. [Be sure you can use a calculator to determine the decimal approximations.] Here's the graph:



EXAMPLE 4: **Graph the ellipse: $16x^2 + y^2 = 16$**

Solution: Is this an ellipse? It doesn't have the same form as the equations in Examples 2 and 3. Let's find the intercepts, then a few other points to construct the graph.

By letting $x = 0$, we get y-values of ±4. By letting $y = 0$, we get x-values of ±1. Therefore, the intercepts are

(1, 0), **(–1, 0)**, **(0, 4)**, and **(0, –4)**

Now we'll let $x = 0.5$. This will give us

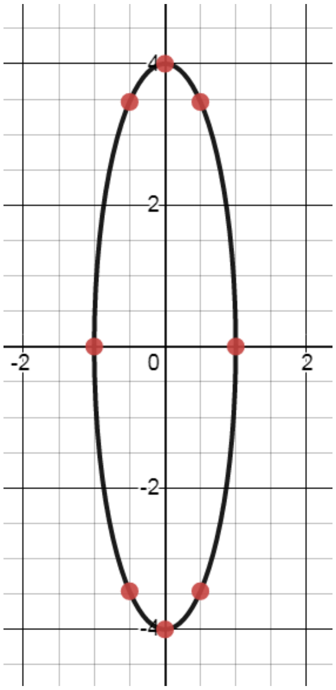
$$16(\mathbf{0.5})^2 + y^2 = 16 \Rightarrow 16(0.25) + y^2 = 16$$

$$\Rightarrow 4 + y^2 = 16 \Rightarrow y^2 = 12 \Rightarrow y = \pm\sqrt{12} \Rightarrow y \approx \pm 3.46$$

Similarly, letting $x = -0.5$ will give the same y-values: ±3.46. We now have four additional points to graph:

(0.5, 3.46), **(0.5, –3.46)**, **(–0.5, 3.46)**, and **(–0.5, –3.46)**

Let's plot all eight of our points — sure enough, we get another ellipse:



If we take the given ellipse

$$16x^2 + y^2 = 16$$

and divide each side by 16, we get

$$\frac{16x^2 + y^2}{16} = \frac{16}{16}$$

$$\frac{x^2}{1} + \frac{y^2}{16} = 1$$

which is the same form as the two previous examples.

Homework

2. Prove that no point with an x-coordinate of 12 will lie on the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

3. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

   a. Find all four intercepts.
   b. Find two more points on the ellipse by letting $x = 2$.
   c. Find two more points on the ellipse by letting $x = -3$.
   d. Sketch the ellipse.

4. Find all four intercepts of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Graph each ellipse by plotting the four intercepts and four additional points:

5. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

6. $x^2 + \frac{y^2}{49} = 1$

7. $x^2 + 9y^2 = 9$

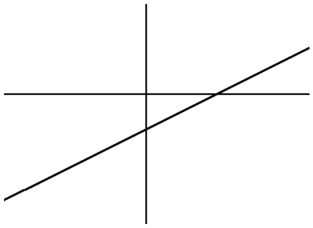
8. $\frac{x^2}{81} + \frac{y^2}{9} = 1$

9. $\frac{x^2}{49} + \frac{y^2}{100} = 1$

10. $25x^2 + 4y^2 = 100$

☐ *LINES, PARABOLAS, CIRCLES, AND ELLIPSES*

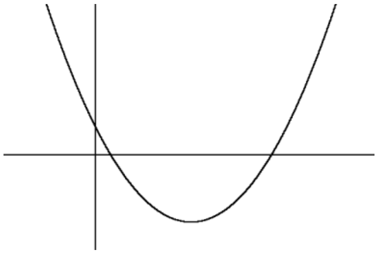
If you've covered lines, parabolas, circles, and ellipses in your course, it's the right time to consider the question: Given an equation of one of these four graphs, how can we tell (without graphing) which shape it will be?



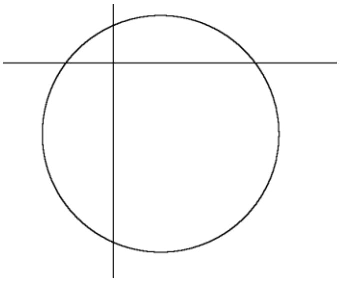
The answer depends primarily on how many of the two variables x and y are squared in the equation. In a ***line***, neither variable is squared; for example,

$$3x - 4y = 10 \qquad x = -3 \qquad y = 7$$

are the equations of lines.

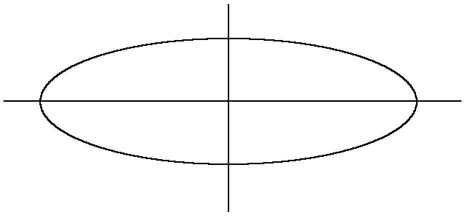


If exactly one variable is squared, then it's a ***parabola***. In a beginning course, a typical parabola might have the equation $y = 2x^2 - 5x + 3$, whose graph opens up. It's also possible that the y is squared, like $x = y^2 + 3$, whose graph is a sideways parabola.

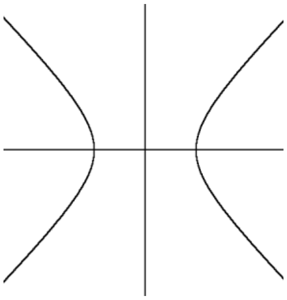


An equation with both variables squared, such as $x^2 + y^2 = 25$, has a graph which is a ***circle*** whose center is the origin and whose radius is 5. The equation $(x - 2)^2 + (y + 7)^2 = 20$ also is a circle, since it's pretty clear that, if expanded, both variables are being squared. In each circle, the coefficients of the two squared terms are the same, namely 1.

As for the ***ellipse***, a typical equation was $9x^2 + 4y^2 = 36$, or perhaps something like $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Both variables are squared in these equations, so why aren't they circles? Here's the answer: In a circle, the coefficients of the squared terms are always the same; in an ellipse, they're <u>different</u> but <u>have the same sign</u>. In the first ellipse formula above, the coefficients are 9 and 4 (both positive); in the second ellipse, the coefficients are $\frac{1}{25}$ and $\frac{1}{4}$ (again, both positive). Note that the



equation $7x^2 + 7y^2 = 30$ is actually a circle because both variables are squared and the coefficients are the same. In fact, if you divide each side of the equation by 7, it might be clearer that it's indeed a circle.



We close the chapter by looking at the equation $x^2 - y^2 = 6$. It's not a line because both variables are squared. It's not a parabola for precisely the same reason. It's not a circle since the coefficients (1 and –1) are different. And it's not an ellipse because the coefficients don't have the same sign. In other words, it's likely a new graph to you and is called a ***hyperbola*** (hy-PER-bo-la),

Homework

11. For each formula, determine whether its graph is a line, a parabola, a circle, or an ellipse.

a. $3x - 4y = 22$

b. $x = -9$

c. $y - 7 = 0$

d. $y = -x^2 + 13x$

e. $x^2 + y^2 = 7$

f. $2x^2 + 3y^2 = 23$

g. $(x - 2)^2 + y^2 = 9$

h. $\frac{x^2}{5} + \frac{y^2}{6} = 1$

i. $\frac{x^2}{23} + \frac{y^2}{2} = 1$

j. $17x^2 + 22y^2 = 44$

k. $3x + y^2 - 6 = 0$

l. $(x + 1)^2 + (y - 2)^2 = 29$

m. $y = 17x - 19$

n. $x^2 - y^2 = 12$

Review Problems

12. Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{100} = 1$.

    a. Prove that $x = 0$ is allowed in the formula.
    b. Prove that $x = 4$ is <u>not</u> allowed in the formula.

13. Find all the intercepts of $25x^2 + 4y^2 = 100$.

14. True/False:

    a. The ellipse $25x^2 + 4y^2 = 100$ has four intercepts.
    b. The graph of $9x^2 - 16y^2 = 144$ is an ellipse.
    c. The graph of $10x^2 = y^2 + 100$ is an ellipse.

15. Matching:

_____	$9x^2 - 18y + 7x + 9y^2 + 1 = 0$	A. line
_____	$18x - \pi y = 171.45$	B. parabola
_____	$y = \sqrt{2}\,x^2 - 200x + \frac{17}{2}$	C. circle
_____	$\frac{x^2}{17}+\frac{y^2}{5} = 1$	D. ellipse
_____	$y = -9$	E. hyperbola

16. Graph the ellipse by plotting the four intercepts and four additional points: $\frac{x^2}{9}+\frac{y^2}{25} = 1$

17. Graph the ellipse by plotting the four intercepts and four additional points: $9x^2 + 16y^2 = 576$

10

Solutions

1. a. Setting $y = 0$ gives $x = 10$. Setting $x = 0$ gives $y = 6$. The intercepts are (10, 0) and (0, 6).

b. (–5, 0) and (0, 20)

c. (3, 0)

d. (0, –4)

e. (0, 0) is both the x- and y-intercept.

f. Setting $y = 0$ yields the quadratic equation $x^2 - x - 6 = 0$, which factors into $(x - 3)(x + 2) = 0$, whose solutions are $x = 3, -2$. The x-intercepts are therefore (3, 0) and (–2, 0). The y-intercept is (0, –6).

g. There are four intercepts: (±11, 0) and (0, ±11).

h. $(\pm 5\sqrt{2}, 0)$ and $(0, \pm 5\sqrt{2})$

i. (±1, 0) and (0, ±3)

j. (±6, 0) and (0, ±1)

k. (±5, 0) and (0, ±9)

l. (±4, 0) and (0, ±2)

m. Setting $y = 0$ yields the quadratic equation $x^2 = 100$, with solutions $x = \pm 10$. But setting $x = 0$ gives $-y^2 = 100$, or $y^2 = -100$, which has no solutions in $\mathbb{R}$. The intercepts are (±10, 0).

n. (0, ±11)

o. (0, 1)

<u>Remember</u>: In this book, every intercept is written as an ordered pair.

2. Let $x = 12$ in the ellipse equation, and you should end up with y^2 equals a negative number, resulting in no solutions in $\mathbb{R}$.

3. a. (4, 0) (–4, 0) (0, 2) (0, –2)

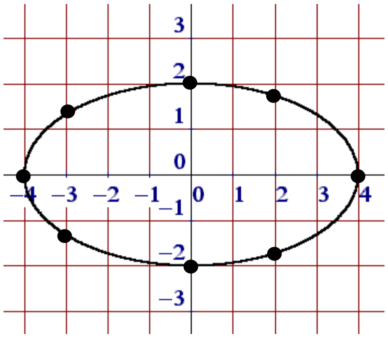
b. Letting $x = 2$ gives

$$\frac{2^2}{16}+\frac{y^2}{4} = 1 \quad\Rightarrow\quad \frac{1}{4}+\frac{y^2}{4} = 1 \quad\Rightarrow\quad \frac{y^2}{4} = \frac{3}{4}$$

$\Rightarrow \quad y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{3}$, which give us the two approximate points (2, 1.732) and (2, –1.732).

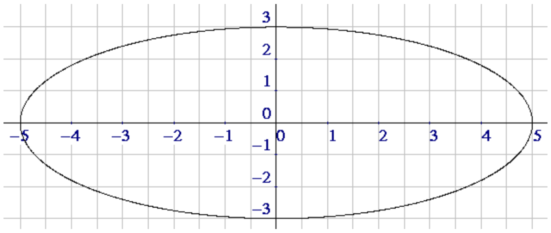
c. Letting $x = -3$ gives the points (–3, 1.323) and (–3, –1.323).

d.

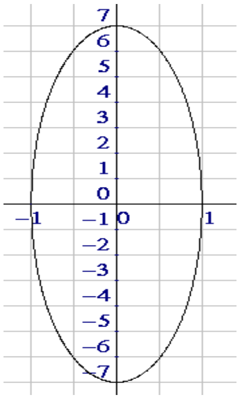


4. $(a, 0)$ $(-a, 0)$ $(0, b)$ $(0, -b)$

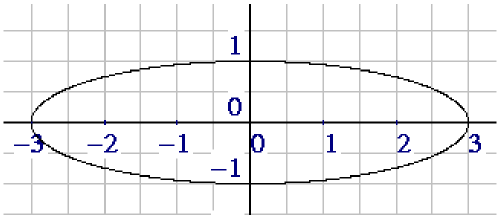
5.



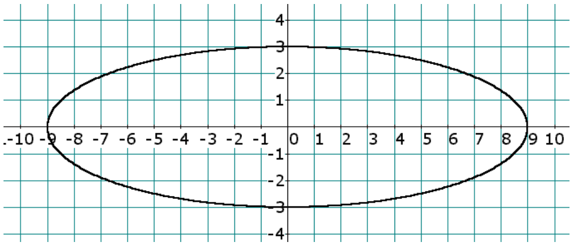
6.



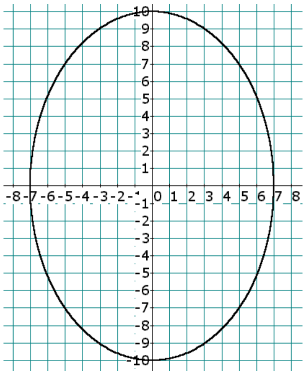
7.



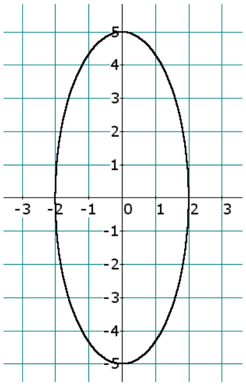
8.



9.



10.



11. a. Neither variable is squared, so it's a line.

b. It's a vertical line.

c. It's a horizontal line.

d. One variable is squared; the other isn't. It's a parabola.

e. Both variables are squared, each with the same coefficient (1), so it's a circle.

f. Both variables are squared, but with different, positive coefficients; therefore, it's an ellipse.

g. Both variables are squared, each with the same coefficient (1). It's a circle.

h. Both variables are squared, and the coefficients are 1/5 and 1/6; thus, it's an ellipse.

i. Both variables are squared, and the coefficients are 1/23 and 1/2; it follows that it's an ellipse.

j. Both variables are squared, and the coefficients are 17 and 22; so it's an ellipse.

k. One variable is squared; the other isn't. It's a parabola.

l. Circle

m. Line

n. Both variables are squared, but their coefficients have opposite signs. So it is neither a line, a parabola, a circle, nor an ellipse; it's a hyperbola.

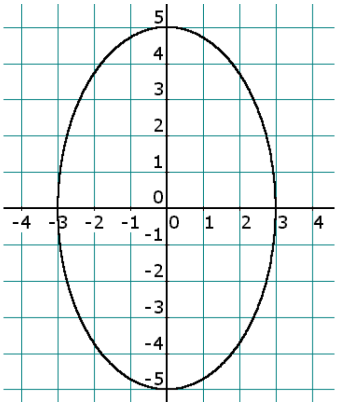
12. a. Letting $x = 0$ in the ellipse equation results in $y = \pm 10$, which of course are real numbers. Therefore, 0 is allowed in the formula.

b. Letting $x = 4$ in the equation results in $y^2 = -\frac{700}{9}$, which has no solution in the real numbers. Thus, 4 is not allowed in the formula.

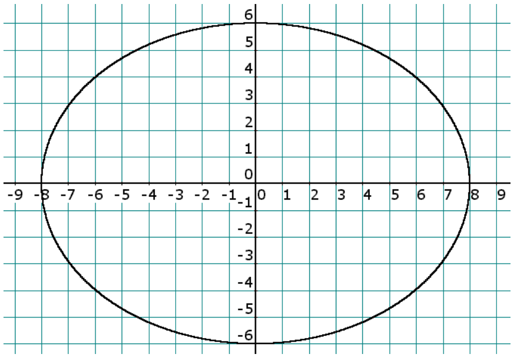
13. (±2, 0) (0 ±5)

14. a. T b. F c. F **15**. C, A, B, D, A

16.



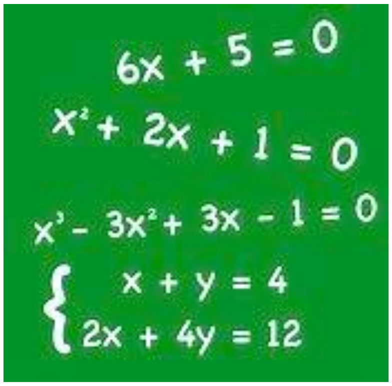
17.



"Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air."

- ***John Quincy Adams***

CH XX – CUBIC AND QUARTIC EQUATIONS



This chapter is a collection of additional equations: Cubic (third degree) and Quartic (fourth degree). Although cubics and quartics can be incredibly difficult to solve, the equations in this chapter can all be solved using the Factoring method. (You can never be too good at factoring!)

❑ *CUBIC EQUATIONS USING THE GCF*

EXAMPLE 1: **Solve the cubic equation: $30x^3 + 99x^2 = 21x$**

Solution: Since factoring is how we solved quadratic equations earlier in the course, let's try factoring here. The factoring method requires one side of the equation to be 0, so we start by subtracting $21x$ from each side of the equation:

$$30x^3 + 99x^2 - 21x = 0$$

Another theme in this chapter is that factoring always begins with an attempt to factor out the GCF, which in this case is $3x$:

$$3x(10x^2 + 33x - 7) = 0$$ [Check by distributing.]

Factoring the trinomial in the parentheses gives the final factorization of the left side of our equation:

$$3x(5x - 1)(2x + 7) = 0$$

Interpreting $3x$ is one of the factors, we have <u>three</u> factors whose product is 0; we therefore set <u>each</u> of the three factors to 0:

$$3x = 0 \quad \underline{\text{or}} \quad 5x - 1 = 0 \quad \underline{\text{or}} \quad 2x + 7 = 0$$

Solving each of these three linear equations gives us three solutions to our cubic equation:

$$\boxed{x = 0, \frac{1}{5}, -\frac{7}{2}}$$

Homework

1. Solve each cubic equation:

a. $x^3 + 3x^2 + 2x = 0$

b. $4n^3 - 18n^2 + 8n = 0$

c. $x^3 = 16x$

d. $3y^3 = -30y^2 - 75y$

e. $a^3 + 9a = 0$

f. $30x^3 + 25x^2 - 30x = 0$

2. Solve for x: $x^2(x + 1)(2x - 3)(x + 7)^3(x^2 - 4)(x^2 - 5x + 6) = 0$

☐ *CUBIC EQUATIONS USING GROUPING*

<u>EXAMPLE 2</u>: **Solve the cubic equation: $x^3 + 5x^2 = 9x + 45$**

<u>Solution</u>: The first step is to bring the terms on the right side of the equation to the left:

$$x^3 + 5x^2 - 9x - 45 = 0$$

By grouping the first two terms, we can factor the GCF in the first pair of terms and the last pair of terms:

$$x^2(x + 5) - 9(x + 5) = 0$$

Pull out the GCF, $x + 5$:

$$(x + 5)(x^2 - 9) = 0 \qquad \text{(partially factored)}$$

Continue by factoring the difference of squares:

$$(x + 5)(x + 3)(x - 3) = 0 \qquad \text{(fully factored)}$$

Set each factor to 0:

$$x + 5 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solving each linear equation gives:

$$x = -5 \quad \underline{\text{or}} \quad x = -3 \quad \underline{\text{or}} \quad x = 3$$

And now we have our three solutions:

$$\boxed{x = -5, -3, 3}$$

❑ QUARTIC EQUATIONS

<u>EXAMPLE 3:</u> **Solve the quartic equation: $x^4 - 26x^2 + 25 = 0$**

<u>Solution:</u> The factoring we learned in the chapter *Advanced Factoring* is exactly what we need here:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow \quad (x^2 - 1)(x^2 - 25) = 0$$

Note: Each factor is a difference of squares.

$$\Rightarrow \quad (x + 1)(x - 1)(x + 5)(x - 5) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad \underline{\text{or}} \quad x - 1 = 0 \quad \underline{\text{or}} \quad x + 5 = 0 \quad \underline{\text{or}} \quad x - 5 = 0$$

$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -5 \quad \text{or} \quad x = 5$$

$$\boxed{x = \pm 1, \pm 5}$$

By the way, a fourth-degree polynomial equation can have at most four solutions, but it may have fewer (see the next two examples).

EXAMPLE 4: **Solve for n: $2n^4 - 15n^2 = 27$**

Solution: The factoring technique requires one side of the equation to be 0, so our first step is to make that happen by subtracting 27 from each side of the equation:

$$2n^4 - 15n^2 - 27 = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n^2 - 9) = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n + 3)(n - 3) = 0$$

$$\Rightarrow \quad 2n^2 + 3 = 0 \ \underline{\text{or}} \ n + 3 = 0 \ \underline{\text{or}} \ n - 3 = 0$$

Let's solve the first equation:

$$2n^2 + 3 = 0 \ \Rightarrow \ 2n^2 = -3 \Rightarrow \ n^2 = -\frac{3}{2} \ \Rightarrow \ n = \pm\sqrt{-\frac{3}{2}}$$

which are numbers that are NOT in the set of real numbers, ℝ. So, assuming your algebra class involves only real numbers, we conclude that the equation $2n^2 + 3 = 0$ has No Solution. The other two equations should be easy for you by now. The final solution is

$$\boxed{x = \pm 3}$$

EXAMPLE 5: **Solve for a: $a^4 = -13a^2 - 36$**

Solution:

$$
\begin{aligned}
& a^4 = -13a^2 - 36 \\
\Rightarrow\quad & a^4 + 13a^2 + 36 = 0 \\
\Rightarrow\quad & (a^2 + 9)(a^2 + 4) = 0 \\
\Rightarrow\quad & a^2 + 9 = 0 \text{ or } a^2 + 4 = 0
\end{aligned}
$$

I hope it's clear that neither of these last two equations has a solution in the real numbers, $\mathbb{R}$. We're done:

$$\boxed{\text{No Solution}}$$

Homework

3. Solve each equation:

a. $x^3 + x^2 - 16x = 16$

b. $n^4 = 13n^2 - 36$

c. $2a^4 - 49a^2 = 25$

d. $x^4 + 17x^2 + 16 = 0$

Review Problems

Solve and check each equation:

4. $30x^3 = 2x^2 + 4x$

5. $n^3 + 5 = 5n^2 + n$

6. $x^4 + 900 = 109x^2$

7. $a^4 + 3a^2 - 4 = 0$

8. $t^4 + 49 + 50t^2 = 0$

9. $w^3 + 2w^2 - 12w - 9 = 0$ [Hard]
Hint: To factor, divide by $w - 3$.

Solutions

1. a. $x = 0, -1, -2$ b. $n = 0, \frac{1}{2}, 4$ c. $x = 0, 4, -4$

d. $y = 0, -5$ e. $a = 0$ f. $x = 0, \frac{2}{3}, -\frac{3}{2}$

2. There are quite a few solutions; you're on your own.

3. a. $x = \pm 4, -1$ b. $n = \pm 2, \pm 3$ c. $a = \pm 5$ d. No solution

4. $x = 0, -\frac{1}{3}, \frac{2}{5}$ **5**. $n = 5, \pm 1$ **6**. $x = \pm 3, \pm 10$

7. $a = \pm 1$ **8**. No solution

9. After you divide the cubic by $w - 3$, the other factor should be quadratic, but it is NOT factorable. So the only solution is $w = 3$.

Edith Hamilton:

> “To be able to be
> caught up into the
> world of thought –
> that is educated.”

CH NN – SQUARE-ROOT EQUATIONS

An equation such as $\sqrt{x-4}-5 = 0$, with the variable inside the square-root sign, is called a ***square-root equation*** (or, in general, a *radical equation*). Just as subtraction in an equation is removed by addition, a square root is removed by squaring.

Here are the major steps to solve a general radical equation:

1. Isolate the radical (if necessary).
2. Raise each side of the equation to an appropriate power (squaring in this chapter).
3. Solve the resulting equation for potential solutions.
4. Check every potential solution in the original equation.

☐ *EXAMPLES*

The key to solving a square-root equation is the fact that, assuming $x \ge 0$,

$$\left(\sqrt{x}\right)^2 = x$$

EXAMPLE 1: **Solve for z: $\sqrt{z} = 7$**

Solution: The radical is already isolated, so on to step 2. Since the radical is a square root, we will square each side of the equation:

$$\sqrt{z} = 7 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad \left(\sqrt{z}\right)^2 = 7^2$$ (square each side of the equation)

$$\Rightarrow \quad \boxed{z = 49}$$ (squaring undoes square rooting)

Put 49 back into the original equation, and you'll see that we have a valid solution.

<u>EXAMPLE 2:</u> **Solve for *x*:** $\sqrt{\boldsymbol{x}-\mathbf{4}}-\mathbf{5} = \mathbf{0}$

<u>Solution:</u> Our plan is to isolate and square.

Start with the original equation: $\sqrt{x-4}-5 = 0$

First, isolate the radical: $\sqrt{x-4} = 5$

Second, to remove the square root sign, square each side of the equation: $\left(\sqrt{x-4}\right)^2 = 5^2$

Simplify each side of the equation: $x - 4 = 25$

Now solve for *x*: $\underline{x = 29}$

Now it's time to check our solution. Let's put our candidate, 29, into the <u>original</u> equation and perform the arithmetic separately on each side of the equation:

$$\begin{array}{l|l} \sqrt{\mathbf{29}-4}-5 & \underline{0} \\ \sqrt{25}-5 & \\ 5-5 & \\ \underline{0} & \checkmark \end{array}$$

So 29 checks out; therefore, the solution is $\boxed{x = 29}$

EXAMPLE 3: **Solve for a:** $\sqrt{a^2+15}+10 = 2$

Solution: We isolate the radical, square each side, solve for a, and then CHECK the solutions.

$\sqrt{a^2+15}+10 = 2$	(the original equation)
$\Rightarrow \sqrt{a^2+15} = -8$	(isolate the radical)
$\Rightarrow \left(\sqrt{a^2+15}\right)^2 = (-8)^2$	(square each side
$\Rightarrow a^2 + 15 = 64$	(simplify each side)
$\Rightarrow a^2 - 49 = 0$	(subtract 64 from each side)
$\Rightarrow (a+7)(a-7) = 0$	(factor)
$\Rightarrow a + 7 = 0 \text{ or } a - 7 = 0$	(set each factor to 0)
$\Rightarrow \underline{a = -7 \text{ or } a = 7}$	(solve for a)

These are two possible solutions, 7 and –7. Let's check out the **7**:

$\sqrt{\mathbf{7}^2+15}+10$	$\underline{2}$
$\sqrt{49+16}+10$	
$\sqrt{64}+10$	
$8 + 10$	
$\underline{18}$	✗

The two sides of the equation do <u>not</u> balance; therefore, 7 is <u>not</u> a solution of the radical equation. Using –7 for a will result in the same calculations, indicating that –7 is also not a solution. So, what do we have here? We solved an equation and got two potential solutions, 7 and –7, but neither of them panned out (they are called *extraneous solutions*). That's it — it's over — nothing works.

$$\boxed{\text{No Solution}}$$

EXAMPLE 4: **Solve for x: $\sqrt{11-x}+5 = x$**

Solution: To isolate the radical, we need to subtract 5 from each side of the equation:

$$\sqrt{11-x} = x-5$$

Since we now have a square root to undo, we will square each side of the equation:

$$\left(\sqrt{11-x}\right)^{2} = (x-5)^{2}$$ Notice the parentheses around the $x - 5$.

Now we simplify each side of the equation:

$$11 - x = x^2 - 10x + 25$$ Careful when squaring $x - 5$.

Recognizing this equation as quadratic, we'll put it into standard form by adding x to each side and subtracting 11 from each side:

$$0 = x^2 - 9x + 14$$

You're welcome to switch the sides and write

$$x^2 - 9x + 14 = 0$$

Factoring produces

$$0 = (x-7)(x-2)$$

Setting each factor to 0 and solving gives us two candidates for a solution:

$\underline{x = 7 \text{ or } x = 2}$

We now check each potential solution in the original equation:

$\underline{x = 7}$

$\sqrt{11-\mathbf{7}}+5$	$\underline{\mathbf{7}}$
$\sqrt{4}+5$	
$2 + 5$	
$\underline{7}$	✓

$\underline{x = 2}$

$\sqrt{11-\mathbf{2}}+5$	$\underline{\mathbf{2}}$
$\sqrt{9}+5$	
$3 + 5$	
$\underline{8}$	✗

What does all this mean? Easy — the 7 works while the 2 doesn't. So, even though we got two solutions when we solved the radical equation, only one actually worked out.

The 2 we obtained but failed to work in the original equation is called an *extraneous solution*. (By the way, just because one solution failed to work does not mean we made any mistakes when solving the problem.) Therefore, the final solution of the equation is

$$\boxed{x = 7}$$

Homework

Solve and check each equation:

1. $\sqrt{x} = 10$
2. $\sqrt{y} = 5$
3. $\sqrt{t} = 0$
4. $\sqrt{a} = -3$
5. $\sqrt{x+1} = 3$
6. $\sqrt{c-5} = 10$
7. $\sqrt{x+5} - 2 = 3$
8. $\sqrt{2y-1} + 3 = 6$
9. $\sqrt{8-n} - 4 = 0$
10. $\sqrt{7z+100} + 10 = 20$
11. $\sqrt{5x-1} = \sqrt{7x-25}$
12. $\sqrt{x} + 2 = x$
13. $\sqrt{n+1} + 3 = n - 8$
14. $\sqrt{2x-11} + x = 13$

☐ *To ∞ and Beyond*

A. Solve for x: $\sqrt[5]{7x-9} = 2$

B. Solve for x: $\sqrt{x+40} = 4 + \sqrt{x}$

C. Solve for y: $\sqrt[3]{(y-2)^2} = 4$

Solutions

1. $x = 100$ **2**. $y = 25$ **3**. $t = 0$ **4**. No solution

5. $x = 8$ **6**. $c = 105$ **7**. $x = 20$ **8**. $y = 5$

9. $n = -8$ **10**. $z = 0$

11. Square each side: $5x - 1 = 7x - 25 \Rightarrow x = 12$; check it in the original equation, doing just simple arithmetic on each side separately. You should find that $x = 12$ works.

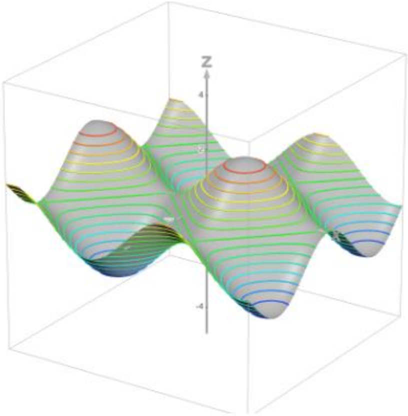
12. Isolate the radical and square each side, leading to the equation $x^2 - 5x + 4 = 0$, whose solutions are 4 and 1. Only the 4 works; the value 1 is called an *extraneous* solution. The final answer is $x = 4$.

13. Isolate the radical and square both sides — when simplified, you'll have the quadratic equation $n^2 - 23n + 120 = 0$. Only $n = 15$ works.

14. The two candidates for a solution are 10 and 18. The 18 is extraneous and the final answer is $x = 10$.

"The only person who is educated is the one who has learned how to learn . . . and change." - Carl Rogers, Psychologist

CH XX – EQUATIONS AND INEQUALITIES VIA DESMOS



Some equations, like $3x - 4 = 7x + 9$, are not too hard to solve. Others can be very difficult, if not impossible, to solve using algebra. For example, I have no idea how to solve the simple-looking equation

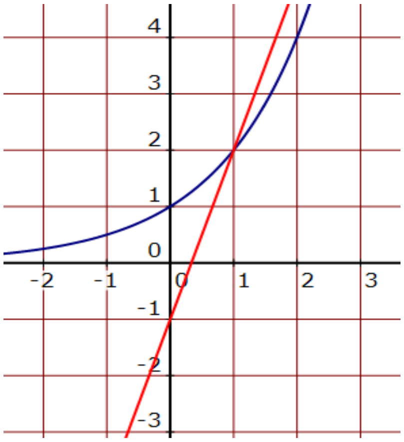
$$2^x = 3x - 1$$

using algebra. But we can find good (if not perfect) solutions using *GRAPHING*, as you'll see in the next example.

☐ *OBTAINING SOLUTIONS FROM GRAPHS*

EXAMPLE 1: **Solve the equation $2^x = 3x - 1$ graphically.**

Solution: I'm going to give you the graph of each side of the equation on the same grid:



From your knowledge of graphing lines, you should see that the red straight line is the graph of $y = 3x - 1$, the right side of the equation. And so the curvy blue graph is the graph of $y = 2^x$, the left side of the equation.

Now, the equation $2^x = 3x - 1$ is a statement of equality — we want to know what values of x will make each side of the equation result in the same number. To do that, we find any ***points of intersection*** of the two graphs. Looking at the graph, a point of intersection seems to be (1, 2). That means that when $x = 1$, both graphs have a y-value of 2, which also implies that when $x = 1$, both sides of the equation are equal (they're both equal to 2). In short, the solution of the equation $2^x = 3x - 1$ (as far as the graph is concerned) is

$$\boxed{x = 1}$$

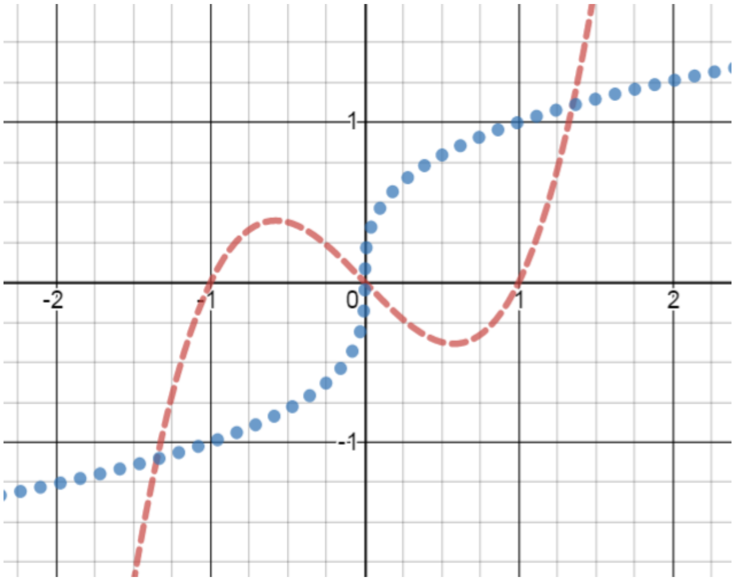
Two Caveats (things to watch out for):

1. How do we know that there's only one solution? Perhaps the two graphs cross somewhere higher up in Quadrant I, or maybe down in Quadrant III — using the graph we have, we can never know. [By simple substitution, you can show that $x = 3$ is another solution of the equation. But are there others?]

2. It may appear that the point of intersection is (1, 2), but a graph is just a rough picture — no graph can be perfectly accurate. So maybe the true value of x at the point of intersection is 0.999997, or maybe 1.0000203. If it IS one of those, the graph will NEVER tell us that. So the graphing method is not an exact method, but if the equation can't be solved using algebra, a graph may be the best we can get — and may be quite useful for applications both inside and outside of mathematics.

EXAMPLE 2: **Solve the equation $x^3 - x = \sqrt[3]{x}$ graphically.**

Solution: For this problem, I'm not going to even bother telling you which graph goes with which side of the equation. Just know that each side of the equation has been graphed. Your job is to pick out any points of intersection (which are, remember, just approximations). The *x*-values of those points of intersection will be the solutions of the equation.



First, do you see three points of intersection? There's one in Quadrant I, there's one in Quadrant III, and it also appears that the origin, (0, 0), is a point of intersection. Second comes the guessing part: The *x*-values of the three points of intersection are approximately –1.3, 0, and 1.3. So these are our best shots at the solutions of the equation:

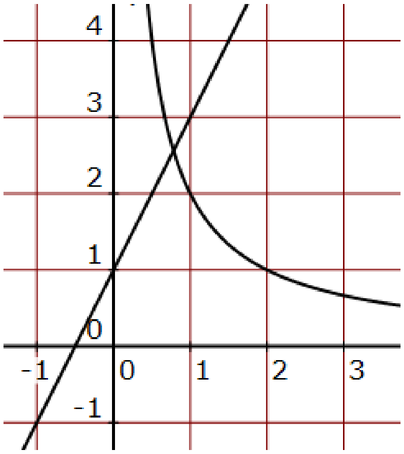
$$\boxed{x = -1.3 \qquad x = 0 \qquad x = 1.3}$$

For the rest of this chapter (after the following homework section), you must graph the given equation yourself before you find the point(s) of intersection.

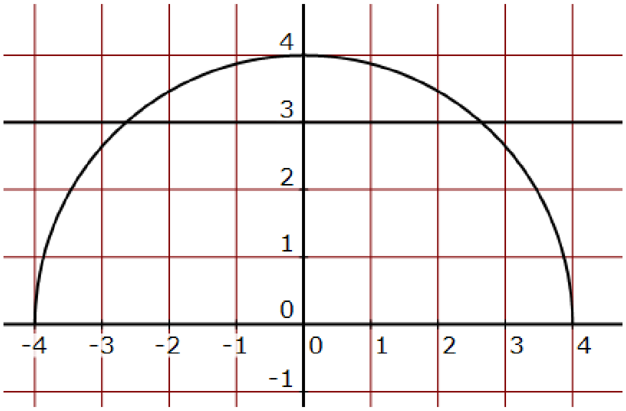
Homework

Solve each equation by using the associated graphs:

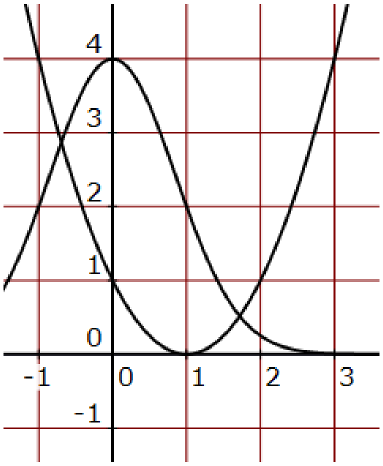
1. $\frac{1}{x} = 2x - 1$



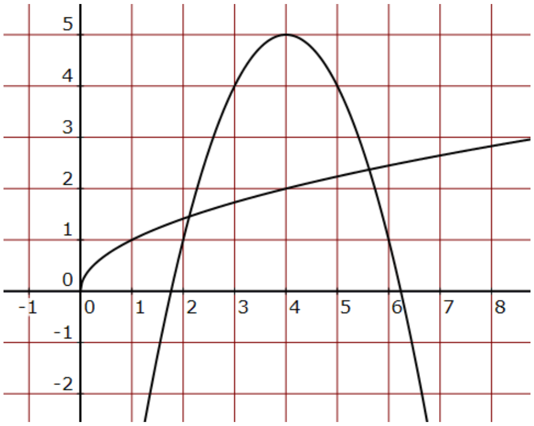
2. $\sqrt{16 - x^2} = 3$



3. $4^{-x^2} = (x-2)^2$



4. $-(x-4)^2 + 5 = \sqrt{x}$

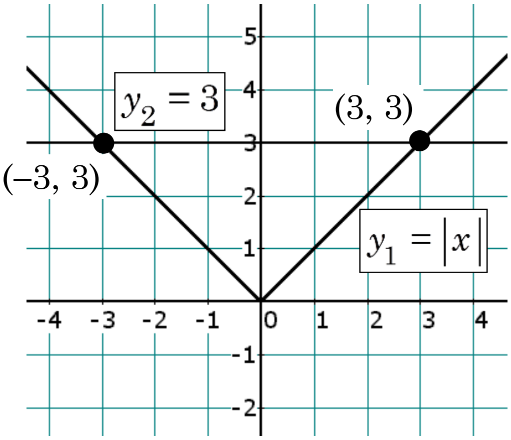


☐ ABSOLUTE VALUE EQUATIONS

EXAMPLE 3: **Solve the equation $|x| = 3$ graphically.**

Solution: We graph each side of the equation separately and look at any points of intersection we might find. To help us keep track, we'll call the left side of the equation y_1 and the right side y_2:

$$y_1 = |x| \qquad y_2 = 3$$



Now use Desmos to graph each equation on the same grid. It appears we have two points of intersection: (–3, 3) and (3, 3). Thus, if $x = -3$ or $x = 3$, the two sides of the equation balance, meaning we have two solutions to our equation:

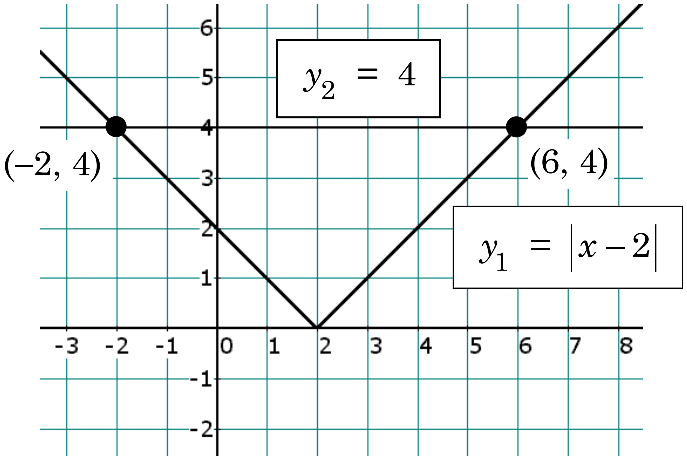
$$\boxed{x = 3 \text{ or } x = -3}$$

[We could also write $x = \pm 3$]

Check:	$x = 3$:	$\lvert x \rvert$	3
		$\lvert 3 \rvert$	
		3	✓
	$x = -3$:	$\lvert x \rvert$	3
		$\lvert -3 \rvert$	
		3	✓

EXAMPLE 4: **Solve the equation $|x-2| = 4$ graphically.**

Solution: Let $y_1 = |x-2|$ and $y_2 = 4$. Graphing y_1 and y_2 on the same grid gives the following graphs:



It looks like y_1 and y_2 intersect at the points (–2, 4) and (6, 4). Therefore, the two solutions of the equation $|x-2| = 4$ are

$$\boxed{x = -2 \text{ or } x = 6}$$

Check:

$x = -2$:

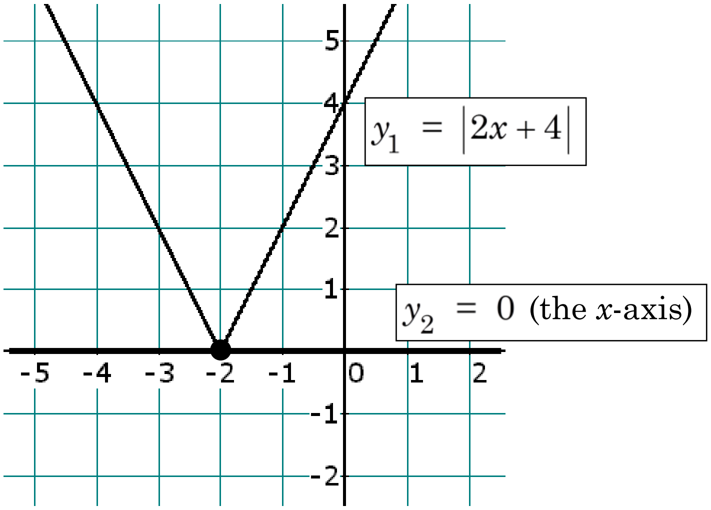
$\|x-2\|$	4
$\|-2-2\|$	
$\|-4\|$	
4	✓

$x = 6$:

$\|x-2\|$	4
$\|6-2\|$	
$\|4\|$	
4	✓

EXAMPLE 5: **Solve the equation $|2x+4| = 0$ graphically.**

Solution: Let $y_1 = |2x+4|$ and $y_2 = 0$. Noting that the graph of $y_2 = 0$ is just the x-axis, we graph y_1 and y_2 on the same grid:



The two graphs intersect at one point, (–2, 0). Thus, each formula has the same value (0) when $x = -2$. Thus, the only solution of the given equation is

$$\boxed{x = -2}$$

Check:

$\|2x+4\|$	0
$\|2(-2)+4\|$	
$\|-4+4\|$	
$\|0\|$	
0	✓

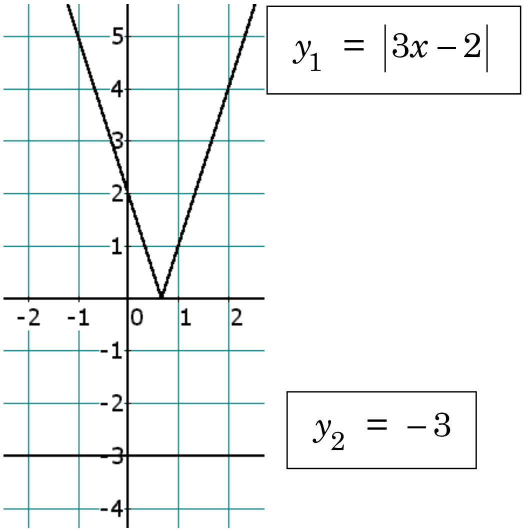
EXAMPLE 6: **Solve the equation $|3x-2| = -3$ graphically.**

Solution: Let's graph $y_1 = 3x-2$ and $y_2 = -3$:

Look at the two graphs. Notice that y_1 and y_2 have no point of intersection. This means that y_1 can never be equal to y_2.

What's our conclusion? The given equation has

No solution



$y_1 = |3x-2|$

$y_2 = -3$

Homework

5. Solve each absolute-value equation graphically:

a. $|x| = 2$ b. $|x| = -3$ c. $|x+1| = 3$

d. $|x-3| = 0$ e. $|2x-2| = 4$ f. $|2x+4| = -1$

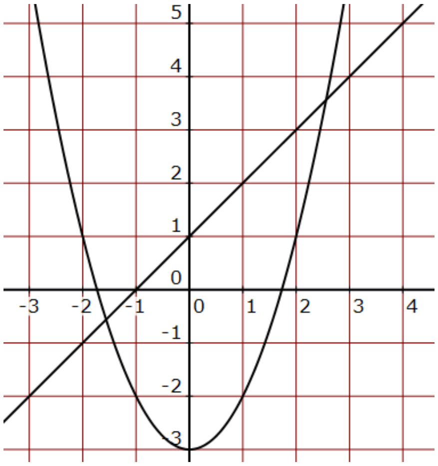
☐ LINES AND CURVES

EXAMPLE 7: **Solve the equation $x^2 - 3 = x + 1$ graphically.**

Solution: Let's graph $y_1 = x^2 - 3$ and $y_2 = x + 1$ on the same grid, and then check out if there are any points of intersection.

The formula $y_2 = x + 1$ is just a straight line, covered in the previous chapter. But the formula $y_1 = x^2 - 3$ might require a little more work. Let's make a table of *x*-*y* values:

x	y_1
–3	6
–2	1
–1	–2
0	–3
1	–2
2	1
3	6



Confirm these graphs with Desmos.

Now we try to determine where the line and the curve intersect. It appears (although there's no guarantee) that the points of intersection are **(–1.5, –0.5)** and **(2.5, 3.5)**. We can therefore estimate the two solutions of the original equation as

$$\boxed{x = -1.5 \text{ or } x = 2.5}$$

Note: The solutions we obtained in this problem are NOT the actual solutions — they are simply the best we can get from the picture. Later in the course, we'll have a couple of ways to find the <u>exact</u> solutions.

Homework

Use DESMOS to approximate the solutions of each equation:

6. $x^2 - 2 = x + 4$

7. $x^2 - 1 = 2x - 2$

8. $x^2 + 2 = x - 1$

Use DESMOS to prove the following assertions:

9. The quadratic equation $2x^2 - 8x + 41 = 0$ has NO solution in $\mathbb{R}$.

10. The quadratic equation $25x^2 - 26x + 6.76 = 0$ has ONE solution in $\mathbb{R}$.

11. The quadratic equation $5x^2 + 97x - 168 = 0$ has TWO solutions in $\mathbb{R}$.

12. The equation $e^x = \ln x$ has NO solution in $\mathbb{R}$.

☐ *INEQUALITIES*

An inequality is quite different from an equation. Whereas an equation usually has one or a couple of solutions, inequalities tend to have infinitely many solutions. Here's an example:

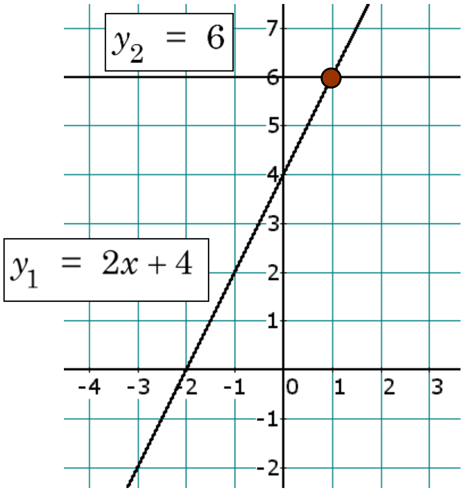
Solve for x: $5x - 3 > 12$

We're seeking any and all values of x which make the statement true. Suppose $x = 6$. Then $5(\mathbf{6}) - 3 = 30 - 3 = 27$, which is > 12. Thus $x = 6$ is a solution of the inequality; but it's "a" solution, not "the" solution. Why? Because there are others; $x = 5$ will work; $x = 3.7$ will work.

But will $x = 3$ work? Let's see: $5(\mathbf{3}) - 3 = 15 - 3 = 12$, which is not greater than 12. So $x = 3$ fails to satisfy the inequality. What about $x = 0$? $5(\mathbf{0}) - 3 = 0 - 3 = -3$, which is certainly not greater than 12. Let's summarize: 0 and 3 failed, but 3.7 and 5 and 6 worked. Our best guess at this point is that any number greater than 3 will work. Our solution is therefore $\boldsymbol{x > 3}$.

EXAMPLE 8: **Solve the inequality $2x + 4 < 6$ graphically.**

Solution: We employ the same technique we used for all the previous equations in this chapter, except we will take into account the "less than" sign after we construct our graphs. Let $y_1 = 2x + 4$ and $y_2 = 6$, both of which are lines.

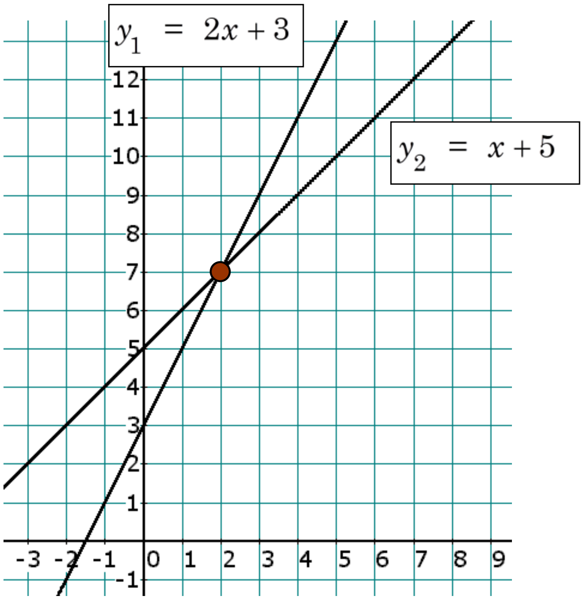


First we note that the two lines appear to intersect at the point (1, 6). But remember, we're trying to solve the inequality $2x + 4 < 6$, so here's the question: For what values of x is the quantity $2x + 4$ smaller than 6? Equivalently — and here's the key question — where on the graph is y_1 below y_2? By looking at the graphs of the lines, we see that y_1 is below y_2 to the left of the point of intersection, (1, 6). In other words, whenever x is smaller than 1, it will follow that $2x + 4$ is less than 6. That is, whenever $x < 1$, we can conclude that $2x + 4 < 6$. We have found the solution of the inequality $2x + 4 < 6$:

$\boxed{x < 1}$ which can be written $(-\infty, 1)$

EXAMPLE 9: **Solve the inequality $2x + 3 \geq x + 5$ graphically.**

Solution: Graph $y_1 = 2x + 3$ and $y_2 = x + 5$ on the same grid.



We're looking for where y_1 is greater than or equal to y_2. Clearly, y_1 and y_2 are equal to each other at the point of intersection, (2, 7). Also, $y_1 > y_2$ wherever the graph of y_1 is above the graph of y_2. This occurs to the <u>right</u> of the point (2, 7); that is, $y_1 > y_2$ whenever $x > 2$. In short, $y_1 \geq y_2$ whenever $x \geq 2$. Thus, the solution of the inequality $2x + 3 \geq x + 5$ is

$\boxed{x \geq 2}$ and in interval notation: $[2, \infty)$

Homework

13. Solve each inequality graphically. When you get tired of graphing, use DESMOS:

a. $x + 3 < 6$

b. $x - 1 \geq 5$

c. $2x + 1 \leq 5$

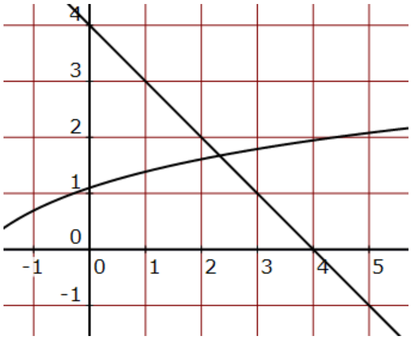
d. $x + 5 \geq 3x - 1$

e. $2x - 1 < x + 2$

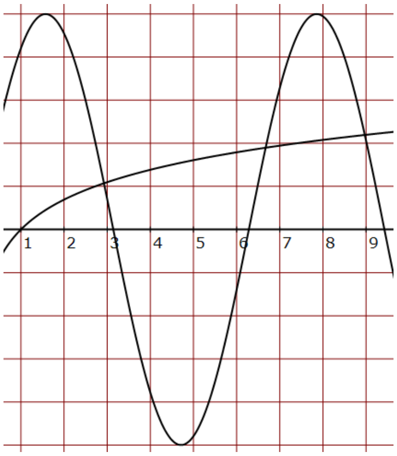
f. $3x + 1 \leq 2x - 3$

Review Problems

14. Solve the equation $\ln(x+3) = -x+4$ by using the associated graphs:



15. Solve the equation $5\sin x = \ln x$ by using the associated graphs:



16. Solve the equation $\left|2x-3\right| = 3$ graphically.

17. Solve the equation $\left|x+1\right|+3 = 3$ graphically.

18. Solve the equation $\left|x-1\right|+2 = 1$ graphically.

19. Solve the equation $x^2 - 4 = 2x - 3$ graphically.

20. Solve the inequality $2x - 2 \le 4$ graphically.

21. Solve the inequality $2x + 3 > x - 2$ graphically.

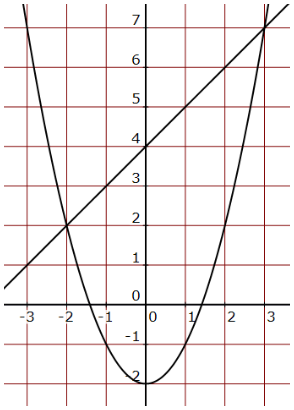
Solutions

Remember that these solutions are approximations; as long as your estimation is close to mine, you have the correct answer.

1. $x = 0.8$ **2.** $x = 2.6, -2.6$ **3.** $x = 1.7, -0.7$ **4.** $x = 2.1, 5.6$

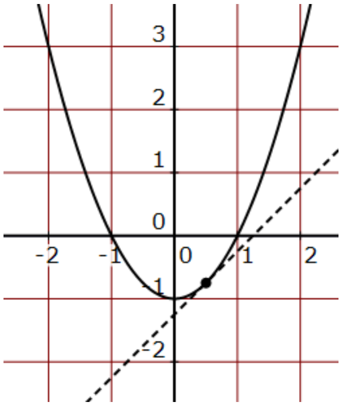
5. a. $x = 2$ or $x = -2$ b. No solution c. $x = 2$ or $x = -4$
d. $x = 3$ e. $x = 3$ or $x = -1$ f. No solution

6.



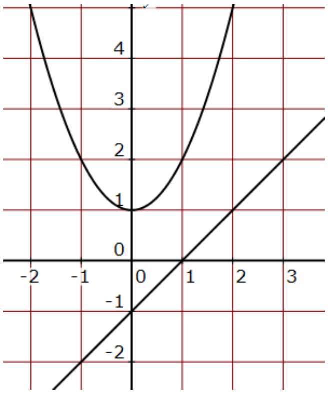
$x = -2, 3$

7.



$x = 0.5$

8.



No Solution

9. **10.** **11.** Graph each side of the equation, noting that $y_2 = 0$ is simply the x-axis. This means that the number of solutions is given by the number of points of intersection between the parabola on the left side of the equation and the x-axis on the right side.

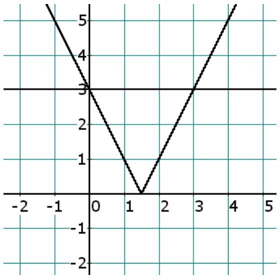
12. The graphs should prove to you that the equation has NO solution.

13. a. $x < 3$ b. $x \geq 6$ c. $x \leq 2$ d. $x \leq 3$
e. $x < 3$ f. $x \leq -4$

14. $x = 2.3$ (or anything reasonably close)

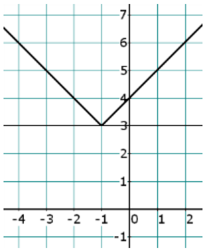
15. There are three points of intersection shown on the graph; the x-values are roughly 2.9, 6.7, and 9.

16.



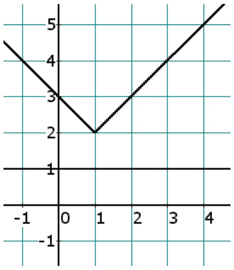
$x = 0, 3$

17.



$x = -1$

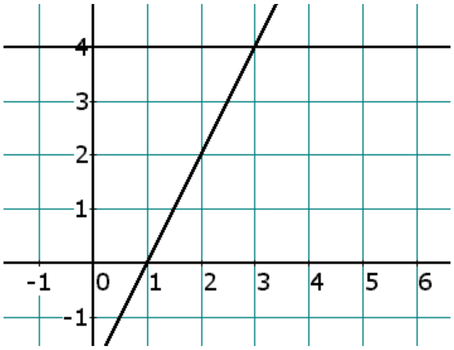
18.



No solution

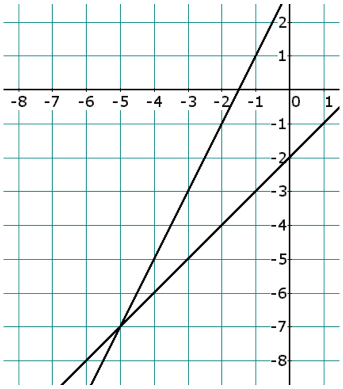
19. $x = -0.4$ or $x = 2.4$ (or anything reasonably close)

20.



$x \leq 3$

21.



$x > -5$

16

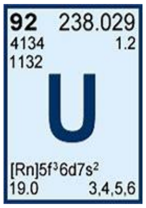
"The first step to getting the things you want out of life is this:

Decide what you want."

- Ben Stein

CH NN – EXPONENTIAL EQUATIONS, NO LOGS

Watching your investments grow, tracking populations, the decaying of a radioactive substance — these are the kinds of problems that lead to exponential equations.



Uranium, atomic number 92, decays exponentially – the less that remains, the less that decays.

☐ *EXAMPLES*

An ***exponential equation*** is an equation with the variable in the exponent, something you might not be used to seeing. Let's start with two basic examples.

#1: Consider the exponential equation

$$5^x = 25$$

Five raised to what power equals 25? Well, 5 to the **2nd** power is 25, so $\boldsymbol{x = 2}$. Not so tough.

#2: How about the equation

$$3^{2n-1} = 81?$$

You might solve this equation by asking yourself, "3 to what power is 81?" Since 3 to the 4th power is 81, it follows that the exponent, $2n - 1$, must be equal to 4.

Writing this last phrase as an equation, we can find the value of n:

$$2n - 1 = 4 \Rightarrow 2n = 5 \Rightarrow \boldsymbol{n = \frac{5}{2}}. \text{ Done!}$$

Check: Placing $n = \frac{5}{2}$ into the equation $3^{2n-1} = 81$ gives

$$3^{2\left(\frac{5}{2}\right)-1} \overset{?}{=} 81$$

$$3^{\cancel{2}\left(\frac{5}{\cancel{2}}\right)-1} \overset{?}{=} 81$$

$$3^{5-1} \overset{?}{=} 81$$

$$3^{4} \overset{?}{=} 81$$

$$81 = 81$$ ✓

EXAMPLE 1: **Solve for x: $3^{7x} = 3^{14}$**

Solution: Each side of the equation is an exponential expression. Notice that the bases are the same (the 3's), so the only way the two sides of the equation can be equal is if the exponents are equal. In other words, $7x$ must equal 14:

$$7x = 14$$

from which we determine that

$$\boxed{x = 2}$$

EXAMPLE 2: **Solve for y: $5^{4y} = 25$**

Solution: We're not as lucky here as we were in Example 1 — the bases are not the same. But maybe we can make them the same. Suppose we think of 25 as 5^2. Then, each side of the equation will have the same base, and we can set the exponents equal to each other to find the value of y. Let's try all of this:

$5^{4y} = 25$ (the original equation)

$\Rightarrow \quad 5^{4y} = 5^2$ (rewrite 25 with a base of 5)

$\Rightarrow \quad 4y = 2$ (set the exponents equal to each other, since the bases are the same)

$\Rightarrow \quad \boxed{y = \frac{1}{2}}$

EXAMPLE 3: **Solve for z: $27^{-4z} = \frac{1}{9}$**

Solution: This equation's terrible! The bases aren't the same, and it contains a fraction. But look at the 27 in the equation — it equals 3^3. And check out the 9 in the denominator — it can be written as 3^2. So maybe we can write everything in terms of the base **3**:

$27^{-4z} = \frac{1}{9}$ (the original equation)

$\Rightarrow \quad (\mathbf{3}^3)^{-4z} = \frac{1}{\mathbf{3}^2}$ (express 27 and 9 as powers of 3)

$\Rightarrow \quad 3^{-12z} = 3^{-2}$ (exponent rules)

$\Rightarrow \quad -12z = -2$ (bases are the same – set exponents equal)

$\Rightarrow \quad \boxed{z = \frac{1}{6}}$ (solve for z)

Homework

Solve each exponential equation:

1. $16^{6p} = 8$
2. $4^{10q} = 16$
3. $125^{3t} = \frac{1}{5}$
4. $16^{6w} = 16$
5. $4^{-2h} = 64$
6. $25^{6b} = \frac{1}{25}$
7. $25^{8w} = 125$
8. $625^{-4x} = \frac{1}{625}$
9. $125^{-10a} = \frac{1}{25}$
10. $16^{3v} = 2$
11. $4^{9r} = \frac{1}{2}$
12. $9^{-4p} = 729$
13. $625^{-3y} = \frac{1}{25}$
14. $81^{9x} = \frac{1}{729}$
15. $25^{-2k} = \frac{1}{25}$
16. $16^{-5x} = 512$
17. $27^{8b} = \frac{1}{729}$
18. $81^{6a} = 3$
19. $81^{-5r} = 81$
20. $625^{6b} = \frac{1}{25}$
21. $27^{-5v} = 3$
22. $81^{-4k} = \frac{1}{9}$
23. $125^{5p} = \frac{1}{25}$
24. $125^{-10d} = \frac{1}{5}$

EXAMPLE 4: **Solve each equation for *x*:**

A. $e^{5x} = e^{20}$

$\Rightarrow 5x = 20 \Rightarrow \boldsymbol{x = 4}$

B. $e^{2x+1} = (e^3)^{x-5}$

$\Rightarrow e^{2x+1} = e^{3x-15} \Rightarrow 2x+1 = 3x-15$

$\Rightarrow -x = -16 \Rightarrow \boldsymbol{x = 16}$

C. $\frac{1}{e} = e^{2x-1}$

$$\Rightarrow\ e^{-1} = e^{2x-1}\ \Rightarrow\ -1 = 2x-1\ \Rightarrow\ \boldsymbol{x = 0}$$

D. $\frac{1}{\sqrt[3]{e}} = \left(\frac{1}{e^2}\right)^x$

$$\Rightarrow\ \frac{1}{e^{1/3}} = \frac{1^x}{(e^2)^x}\ \Rightarrow\ \frac{1}{e^{1/3}} = \frac{1}{e^{2x}}$$

$$\Rightarrow\ \frac{1}{3} = 2x\ \Rightarrow\ \boldsymbol{x = \frac{1}{6}}$$

EXAMPLE 5: **Solve for x: $3^x = 5$**

Solution: Now we're really up a stump! How can we make the bases the same? We can't, so our current method for solving exponential equations dies at this point. In fact, the solution to this equation will not be a rational number as in the previous four examples, so the best we can do is approximate it. Here's one way to solve the problem, using a calculator.

This is simply a guess-and-check method. Clearly, the solution x of the equation $3^x = 5$ is bigger than 1, since $3^1 = 3$; but x must be smaller than 2, since $3^2 = 9$. So the solution is between 1 and 2:

$$1 < x < 2$$

Now try 1.5. $3^{1.5} = 5.196$, a little too big. Let $x = 1.4$; then $3^{1.4} = 4.656$, a little too small. Thus, we've narrowed the solution to a number between 1.4 and 1.5:

$$1.4 < x < 1.5$$

Since 1.45 is halfway between 1.4 and 1.5, let's give it a try. $3^{1.45} = 4.918$, which is really close to 5. Consider 1.46: $3^{1.46} = 4.973$. Now nudge x up to 1.47: $3^{1.47} = 5.028$, a little too big. So the value of x is between 1.46 and 1.47:

$$1.46 < x < 1.47$$

This is getting boring, but I hope you're getting the point. Let's just take the average of 1.46 and 1.47, and figure we're close enough.

Our best guess is therefore

$$\boxed{x = 1.465}$$

In fact, $3^{1.465} = 5.0001$, so we have a really accurate value of *x*. The following chart is a summary of the calculations we made in this problem:

x	1	1.4	1.45	1.46	1.465	1.47	1.5	2
3^x	3	4.656	4.918	4.973	5.0001	5.028	5.196	9

In the next chapter we will study what are called "log" functions. Then we'll have a third (and much better) way of solving any kind of exponential equation, even if the base is *e*.

Homework

Solve each equation for exact solutions:

25. $2^{7x+1} = 2^{3x}$
26. $6^{8y-1} = 36$
27. $8^{-6z} = \frac{1}{2}$
28. $\frac{1}{25^x} = 125^{3x-1}$
29. $7^{3x} = 7^{5x}$
30. $81^{4-3x} = \sqrt{729}$
31. $e^{2x+1} = e$
32. $\frac{1}{e^2} = e^{1-7x}$
33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e}$

Use your calculator to approximate the solution of each equation:

34. $2^x = 6$　　35. $10^y = 50$　　36. $e^z = 12$

37. Solve by inspection (this means stare at it and guess): $8^x = 5^x$

Solutions

1. $p = \frac{1}{8}$

2. $q = \frac{1}{5}$

3. $t = -\frac{1}{9}$

4. $w = \frac{1}{6}$

5. $h = -\frac{3}{2}$

6. $b = -\frac{1}{6}$

7. $w = \frac{3}{16}$

8. $x = \frac{1}{4}$

9. $a = \frac{1}{15}$

10. $v = \frac{1}{12}$

11. $r = -\frac{1}{18}$

12. $p = -\frac{3}{4}$

13. $y = \frac{1}{6}$

14. $x = -\frac{1}{6}$

15. $k = \frac{1}{2}$

16. $x = -\frac{9}{20}$

17. $b = -\frac{1}{4}$

18. $a = \frac{1}{24}$

19. $r = -\frac{1}{5}$

20. $b = -\frac{1}{12}$

21. $v = -\frac{1}{15}$

22. $k = \frac{1}{8}$

23. $p = -\frac{2}{15}$

24. $d = \frac{1}{30}$

25. $-\frac{1}{4}$

26. $6^{8y-1} = 6^2 \Rightarrow 8y - 1 = 2 \Rightarrow y = \frac{3}{8}$

27. $8^{-6z} = \frac{1}{2} \Rightarrow (2^3)^{-6z} = 2^{-1} \Rightarrow 2^{-18z} = 2^{-1}$

$\Rightarrow -18z = -1 \Rightarrow z = \frac{1}{18}$

28. $\frac{3}{11}$

29. 0

30. $\frac{13}{12}$

31. 0

32. $\frac{3}{7}$

33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e} \Rightarrow \left(e^{-1/2}\right)^x = e^{1/4} \Rightarrow e^{-\frac{1}{2}x} = e^{\frac{1}{4}}$

$\Rightarrow -\frac{1}{2}x = \frac{1}{4} \Rightarrow x = -\frac{1}{2}$

34. 2.58 **35**. 1.7 **36**. 2.48

37. Hint: The solution is the only number with no reciprocal.

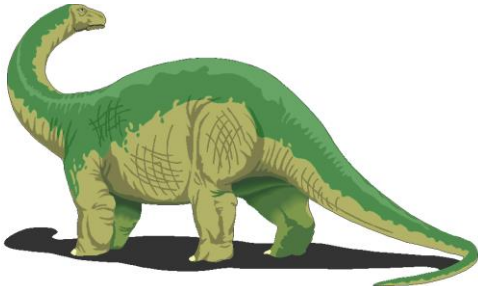
"*Education* is not merely a means for earning a living or an instrument for the acquisition of wealth. It is an initiation into life of spirit, a training of the human soul in the pursuit of truth, and the practice of virtue."

– Vijaya Lakshmi Pandit

CH XX – EXPONENTIAL EQUATIONS, WITH LOGS

When we studied the growth and decay formula a while back, we found that we had no way to determine the growth rate or the amount of time. The chapter *The Laws of Logs* will give us the necessary tools.

$$A = A_0 e^{kt}$$



The exponential decay of radioactive elements allows us to determine the age of dinosaur bones.

EXPONENTIAL EQUATIONS

Remember the hassles in the chapter we encountered in trying to solve the exponential equation $3^x = 5$? We had to use a calculator to guess an answer, and then adjust and guess again, and so on, until we had a couple of digits of accuracy. We're now ready to solve this problem more efficiently.

EXAMPLE 1: **Solve for *x*: $3^x = 5$**

Solution: The variable is in the exponent. This is a dilemma — how do we get the unknown out of the exponent so that we can solve for it? Do you recall the third law of logs? It states that

$$\log_b a^x = x \log_b a$$

It allows us to move the exponent to the front (making it a coefficient), but only if we're taking the log of an expression. So the procedure here will be to take a log (we'll choose *ln*, since it's on your calculator and ln is used in calculus), bring down the exponent, and then solve for it.

$$3^x = 5 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad \mathbf{ln}\ 3^x = \mathbf{ln}\ 5 \qquad \text{(take the } ln \text{ of both sides)}$$

$$\Rightarrow \quad x \ln 3 = \ln 5 \qquad \text{(the third law of logs)}$$

$$\Rightarrow \quad x = \frac{\ln 5}{\ln 3} \qquad \text{(simple algebra -- solve for } x\text{)}$$

$$\Rightarrow \quad x = \frac{1.609437912}{1.098612289} \qquad \text{(use your calculator)}$$

$$\Rightarrow \quad \boxed{x = 1.464973521}$$

When we did this problem by guessing with a calculator, a lot of work produced a guess of 1.465. Clearly, our new method is superior. Note that the true answer, $x = \frac{\ln 5}{\ln 3}$, is an irrational number, while the answer in the box is a rational approximation of the true answer.

EXAMPLE 2: **Solve for n: $2^{3n+2} = 7$**

Solution:

$$2^{3n+2} = 7 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad \mathbf{ln}\, 2^{3n+2} = \mathbf{ln} 7 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad (3n+2)\ln 2 = \ln 7 \qquad \text{(third law of logs)}$$

[Notice the parentheses around the $3n + 2$]

$\Rightarrow$ $(3\ln 2)\,n + 2\ln 2 = \ln 7$ (distribute)

$\Rightarrow$ $(3\ln 2)n = \ln 7 - 2\ln 2$ (subtract 2 ln2)

$\Rightarrow$ $n = \dfrac{\ln 7 - 2\ln 2}{3\ln 2}$ (divide each side by 3 ln 2 to get the **exact** answer)

$\Rightarrow$ $\boxed{n = 0.269118307}$ (use your calculator to get a rational **approximation**)

EXAMPLE 3: **Solve for *a*: $5^{2a-3} = 6^{a+1}$**

Solution:

$5^{2a-3} = 6^{a+1}$ (the original equation)

$\Rightarrow$ $\mathbf{ln}\,5^{2a-3} = \mathbf{ln}6^{a+1}$ (take the *ln* of each side)

$\Rightarrow$ $(2a-3)\ln 5 = (a+1)\ln 6$ (third law of logs)

$\Rightarrow$ $2a\ln 5 - 3\ln 5 = a\ln 6 + \ln 6$ (distribute)

$\Rightarrow$ $2a\ln 5 - a\ln 6 = \ln 6 + 3\ln 5$ (variables to the left and constants to the right)

$\Rightarrow$ $a(2\ln 5 - \ln 6) = \ln 6 + 3\ln 5$ (factor out the variable)

$\Rightarrow$ $\boxed{a = \dfrac{\ln 6 + 3\ln 5}{2\ln 5 - \ln 6}}$ (divide to isolate the a)

This is the exact answer. A rational approximation would be $a = 4.638776075$.

Homework

Solve each equation and round your answers to the nearest ten thousandths place:

1. $2^x = 72$
2. $5^{-y} = 3$
3. $3^{4n-1} = 5$
4. $3^{z+1} = 8^{3-z}$
5. $3^{5c} = 7^{10c}$
6. $e^{3x+4} = 25$
7. $3^{-n} = 43$
8. $7^{4-3x} = 2$
9. $e^x = 2^{x-6}$

❑ THE GROWTH AND DECAY FORMULA REVISITED

The growth and decay formula

$$A = A_0 e^{kt}$$

worked just fine before, when we were searching for either the starting amount, A_0, or the final amount, A. But when the unknown was in the exponent (the k or the t), we were stuck. Now we're not stuck.

EXAMPLE 4: **Assuming an initial population of 7500, a final population of 12,000, and a time period of 7 years, find the annual growth rate.**

Solution: We will write the growth formula, substitute the given values, and then solve for the unknown k:

$$\begin{array}{lll} & A = A_0 e^{kt} & \text{(the growth formula)} \\ \Rightarrow & 12{,}000 = 7500 e^{k \cdot 7} & \text{(substitute the given values)} \\ \Rightarrow & e^{7k} = \dfrac{12{,}000}{7500} & \text{(isolate the } e^{7k}\text{)} \\ \Rightarrow & e^{7k} = 1.6 & \text{(calculator)} \end{array}$$

$$\Rightarrow \quad \mathbf{ln}\, e^{7k} = \mathbf{ln}\, 1.6 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad 7k \ln e = \ln 1.6 \qquad \text{(the third law of logs)}$$

$$\Rightarrow \quad 7k = \ln 1.6 \qquad (\ln e = 1)$$

$$\Rightarrow \quad k = \frac{\ln 1.6}{7} \qquad \text{(solve for } k\text{)}$$

$$\Rightarrow \quad k = 0.067 \qquad \text{(calculator gives a decimal)}$$

And therefore the annual growth rate is

$$\boxed{6.7\%}$$

EXAMPLE 5: **How long will it take for an investment of \$10,000 to reach a total of \$32,000 if the interest rate is 7.3% per year compounded continuously?**

Solution: The phrase "compounded continuously" justifies the use of our growth formula.

$$A = A_0 e^{kt} \qquad \text{(the growth formula)}$$

$$\Rightarrow \quad 32{,}000 = 10{,}000 e^{0.073t} \qquad \text{(remember: } 7.3\% = .073\text{)}$$

$$\Rightarrow \quad \frac{32{,}000}{10{,}000} = \frac{10{,}000 e^{0.073t}}{10{,}000} \qquad \text{(divide each side by 10,000)}$$

$$\Rightarrow \quad e^{0.073t} = 3.2 \qquad \text{(simplify and reverse)}$$

$$\Rightarrow \quad \ln e^{0.073t} = \ln 3.2 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad 0.073t = \ln 3.2 \qquad \text{(third law of logs \& } \ln e = 1\text{)}$$

$$\Rightarrow \quad t = \frac{\ln 3.2}{0.073} = 15.933 \qquad \text{(solve for } t\text{)}$$

Thus, the amount of time it will take to reach the goal is

$$\boxed{15.933 \text{ years}}$$

EXAMPLE 6: **A population is growing at 13% per year, compounded continuously. How long will it take for the population to triple?**

Solution: It would appear that the answer should depend on the starting or ending population. But an interesting aspect of the exponential growth/decay formula is that it doesn't matter. If it takes 10 years for the population to triple from 12 to 36, then it takes the same 10 years for the population to triple from 5,000 to 15,000. The following calculations should prove this.

$$A = A_0 e^{kt}$$

$$36 = 12e^{0.13t} \qquad\qquad 15{,}000 = 5000e^{0.13t}$$

In each case we divide by the initial population:

$$\frac{36}{12} = e^{0.13t} \qquad\qquad \frac{15{,}000}{5000} = e^{0.13t}$$

$$3 = e^{0.13t} \qquad\qquad 3 = e^{0.13t}$$

Note that under either scenario we arrived at exactly the same equation:

$$\begin{aligned} & e^{0.13t} = 3 \\ \Rightarrow\ & \ln e^{0.13t} = \ln 3 \\ \Rightarrow\ & 0.13t = \ln 3 \\ \Rightarrow\ & t = \frac{\ln 3}{0.13} = 8.45 \end{aligned}$$

Therefore, the amount of time needed for the population to triple, regardless of the actual populations, is approximately

$$\boxed{8.45 \text{ years}}$$

Half-life The ***half-life*** of a radioactive substance is the time required for **half** of the material to disintegrate into energy and other particles, which therefore means it's the time it takes for the final amount of the substance to be half of the starting amount. For instance, start with 100 grams of plutonium, and assume the half-life is 7 years. After 7 years, 50 g remain. After another 7 years, 25 g remain. Another 7 years and $12\frac{1}{2}$ g remain, etc., etc. The concept of half-life also pertains to the time it takes for the amount of medication remaining in the body to be half of its starting amount.

EXAMPLE 7: **Find the half-life of a radioactive substance which decays at an annual rate of 9%.**

Solution: Recall the preceding example on tripling the population. It didn't matter whether we went from 12 to 36 or from 5,000 to 15,000 — the tripling time was the same either way. In the same manner, the half-life of a substance does not depend on the starting and ending amounts, as long as the ending amount is half of the starting amount. We can prove this general statement by using variables instead of numbers for the starting and ending amounts.

Let's assume a starting amount of A_0 grams. When half of it decays, the remaining amount is half of A_0, or $\frac{1}{2}A_0$. In other words, $A = \frac{1}{2}A_0$.

$$A = A_0e^{kt} \qquad \text{(the growth/decay formula)}$$

$$\Rightarrow \quad \frac{1}{2}A_0 = A_0e^{-0.09t} \qquad \text{(note: } k \text{ is negative)}$$

$$\Rightarrow \quad \frac{1}{2} = e^{-0.09t} \qquad \text{(divide both sides by } A_0\text{)}$$

$$\Rightarrow \quad \mathbf{ln}\frac{1}{2} = \mathbf{ln}\,e^{-0.09t} \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad -0.09t = \ln\tfrac{1}{2} \qquad \text{(simplify and flip the equation)}$$

$$\Rightarrow \quad t = \frac{\ln\frac{1}{2}}{-0.09} = 7.7$$

And thus the half-life of the substance is about

$$\boxed{7.7 \text{ years}}$$

Homework

10. In the growth formula $A = A_0e^{kt}$, solve for

    a. A_0 b. k c. t

11. Find the interest rate if an investment of \$7200 reached a total of \$18,000 in 5 years.

12. If the population is growing 9% per year, how long will it take for a population of 25,600 to reach a population of 100,000?

13. How long will it take for a population to double in size if the growth rate is 18% per year?

14. Find the half-life of a radioactive substance whose annual decay rate is 23%.

15. At an annual rate of 8%, how many years will it take for you to quadruple your money?

16. The decay rate of a radioactive substance is k (where $k < 0$). Prove that the half-life of the substance is $t = \frac{-\ln 2}{k}$.

Review Problems

17. Solve for a: $5^a = 3$

18. Solve for x: $T = T_0e^{ax}$

19. Solve for x: $7^{3x-1} = 2^{6-5x}$

20. Solve for n: $2^{2n+3} = 3^{4-5n}$

21. Solve for b: $e^{3b} = 2^{b+1}$

22. Solve for x: $10^{7x-14} = 1$

23. Find the annual growth rate if a population increased from 2000 to 7,500 in a period of 9 years.

24. $25,000 is invested in a money market account paying 9.5% per year compounded continuously. How many years will it take for that investment to reach a total of $75,000?

25. How long would it take a population to quintuple in size if the growth rate is 12.7% per year?

26. Find the half-life of a substance whose annual decay rate is 6.3%.

27. True/False:

    a. The solution of the exponential equation $6^x = 4$ is about 0.774.

    b. The exact solution of the equation $2^{3x+2} - 7 = 0$ is $\frac{\ln 1.75}{\ln 8}$.

    c. If it took 12 years for the population to increase from 500 to 2,250, then the annual growth rate was 20%.

    d. If the amount in the 7%/year continuous compounding savings account went from $7,500 to $14,000, then the time span was about 8.916 years.

e. A population is growing at 23% per year compounded continuously. It should take about 1.76 years for the population to grow 50%.

f. The half-life of a substance which decays at an annual rate of 1.3% is about 35 years.

g. If $A = A_0e^{kt}$, then $t = \frac{1}{k}\ln\frac{A}{A_0}$.

Solutions

1. 6.1699 **2.** –0.6826

3. $3^{4n-1} = 5 \Rightarrow \ln(3^{4n-1}) = \ln 5 \Rightarrow (4n-1)\ln 3 = \ln 5$

$$\Rightarrow n = \frac{\frac{\ln 5}{\ln 3}+1}{4} \approx .6162$$

4. 1.6173

5. $3^{5c} = 7^{10c} \Rightarrow \ln(3^{5c}) = \ln(7^{10c}) \Rightarrow 5c\ln 3 = 10c\ln 7$

$$\Rightarrow 5c\ln 3 - 10c\ln 7 = 0 \Rightarrow c(5\ln 3 - 10\ln 7) = 0$$

$$\Rightarrow c = \frac{0}{5\ln 3 - 10\ln 7} \Rightarrow c = 0$$

6. –0.2604 **7.** –3.4236 **8.** 1.2146 **9.** –13.5533

10. a. $A_0 = \frac{A}{e^{kt}}$

b. $A = A_0e^{kt} \Rightarrow \frac{A}{A_0} = e^{kt} \Rightarrow \ln\frac{A}{A_0} = \ln e^{kt}$

$$\Rightarrow \ln A - \ln A_0 = kt \Rightarrow k = \frac{\ln A - \ln A_0}{t}$$

c. $t = \dfrac{\ln A - \ln A_0}{k}$

11. 18% **12**. 15 yrs **13**. 3.9 yrs **14**. 3 yrs **15**. 17.3 yrs

16. $A = A_0e^{kt}$. We assume that the final amount is half of the starting amount: $A = \frac{1}{2}A_0$. Also assume that $k < 0$, since it's a decay problem and not a growth problem.

$$\tfrac{1}{2}A_0 = A_0e^{kt} \Rightarrow \ln\tfrac{1}{2} = \ln e^{kt} \Rightarrow kt = \ln 1 - \ln 2$$

$$\Rightarrow t = \frac{0 - \ln 2}{k} \Rightarrow t = \frac{-\ln 2}{k} \qquad \text{Q.E.D.}$$

17. $a = \dfrac{\ln 3}{\ln 5} \approx 0.6826$

18. $x = \dfrac{\ln T - \ln T_0}{a}$

19. $x = \dfrac{6\ln 2 + \ln 7}{3\ln 7 + 5\ln 2} \approx 0.6562$

20. $n = \dfrac{4\ln 3 - 3\ln 2}{2\ln 2 + 5\ln 3} \approx 0.3365$

21. $b = \dfrac{\ln 2}{3 - \ln 2} \approx 0.3005$

22. $x = 2$

23. $k = 14.7\%$ **24**. 11.56 yrs **25**. 12.67 yrs **26**. 11 yrs

27. a. T b. T c. F d. T e. T f. F g. T

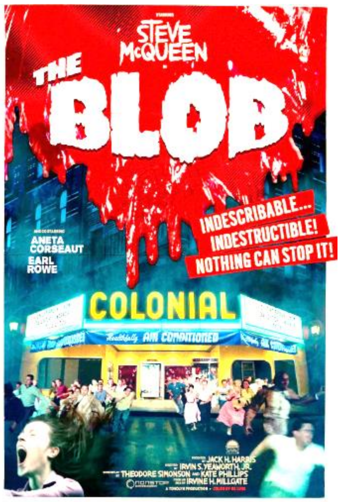
12

It's better to light a candle
than to curse the darkness.

Chinese proverb

CH 58 – EXPONENTIAL FUNCTIONS

What worried Steve McQueen was not that The Blob was growing by a constant amount every hour, but rather that it was **doubling** in size every hour. If our hero had known his math, he could have warned the town that The Blob was not growing linearly, but *exponentially*.



☐ *EXPONENTIAL FUNCTIONS*

EXAMPLE 1: **Analyze The Blob function.**

Solution: To give you an idea of the concept of an exponential function, let's look at two scenarios regarding the rate of growth of The Blob. An example of a linear rate of growth might be the formula $B = 4t$, where t is the time in hours and B is the size of The Blob. An exponential formula could be $B = 2^t$. If we construct a table, showing the time and the amount of The Blob for each formula, we can see the true effect of exponential growth.

t	1	2	3	4	5	6	7	8	9	10	11
$4t$	4	8	12	16	20	24	28	32	36	40	44
2^t	2	4	8	16	32	64	128	256	512	1,024	2,048

For the first three hours, there's more blob in the linear formula than in the exponential formula. At $t = 4$, the blob amounts are equal. But after that, it's not even a contest — the exponential formula shows that The Blob will probably eat the town, the state, and eventually the entire Earth!

EXAMPLE 2: **Find some ordered pairs for the exponential function $f(x) = 4^x$.**

Solution:

$$\begin{aligned}
f(1) &= 4^1 = 4 && \Rightarrow && (1, 4) \\
f(-2) &= 4^{-2} = \frac{1}{4^2} = \frac{1}{16} && \Rightarrow && (-2, \tfrac{1}{16}) \\
f(-1) &= 4^{-1} = \frac{1}{4} && \Rightarrow && (-1, \tfrac{1}{4}) \\
f(0) &= 4^0 = 1 && \Rightarrow && (0, 1) \\
f(\tfrac{1}{2}) &= 4^{1/2} = \sqrt{4} = 2 && \Rightarrow && (\tfrac{1}{2}, 2) \\
f(2) &= 4^2 = 16 && \Rightarrow && (2, 16)
\end{aligned}$$

We now try to determine exactly what the formula for an exponential function looks like, and how it differs from that of a polynomial. Look at the exponential functions given in the previous examples:

$$B = 2^t \qquad f(x) = 4^x$$

Notice that in each function, the base is a constant (the 2 and the 4) and the exponent is a variable (the t and the x). Thus, an ***exponential function*** is a function of the form

$$f(x) = \underset{\substack{\uparrow \\ \boxed{\text{constant}}}}{b}^{x} \leftarrow \boxed{\text{variable}}$$

where b is some appropriate real number (a constant)

This is in sharp contrast to the notion of a *polynomial*, which is the other way around. Thus,

$y = 10^x$ is an *exponential* function,

$y = x^{10}$ is a *polynomial* function, and

$y = x^x$ is neither an exponential nor a polynomial function.

The question of which real numbers ***b*** serve nicely as the base of an exponential function will be discussed later.

Homework

1. a. Fill in the following chart, similar to Example 1:

t	1	2	3	4	5	6	7	8	9	10
$9t$										
3^t										

b. Is the $9t$ row of the chart a linear or exponential function?

c. Is the 3^t row of the chart a linear or exponential function?

d. For how many hours does the linear growth produce more blob than the exponential growth?

e. At what hour do both growths give the same amount of blob?

f. At 10 hours, what is the ratio of the exponential amount of blob to the linear amount of blob?

2. Let's find some more ordered pairs in the function $f(x) = 4^x$ from Example 3.

a. (3, ___) b. (–3, ___) c. $(-\frac{1}{2}, \_\_\_)$ d. $(\frac{3}{2}, \_\_\_)$

3. Take a guess what the domain of $f(x) = 4^x$ is.

❑ GRAPHING EXPONENTIAL FUNCTIONS

Let's use the function $y = 4^x$ discussed in the previous section to make our first exponential graph.

EXAMPLE 3: **Graph: $y = 4^x$**

Solution: Here are some ordered pairs for this function that we found in Example 2 and Homework #2:

$(-1, \frac{1}{4})$ $(-\frac{1}{2}, \frac{1}{2})$ $(0, 1)$ $(\frac{1}{2}, 2)$ $(1, 4)$ $(2, 16)$

Notice that the point (0, 1) is the ***y*-intercept**.

Next we analyze a pair of **limits**. First we'll let $x \to \infty$. As it does, the functional value 4^x approaches ∞ much faster than x does. For example, as x takes the values 6, 8, 10, the y-values go 4096, 65536, 1048576. Wow! The functional values are growing like crazy. The following limit should now be clear:

As $x \to \infty$, $y \to \infty$ [an amazingly fast increase]

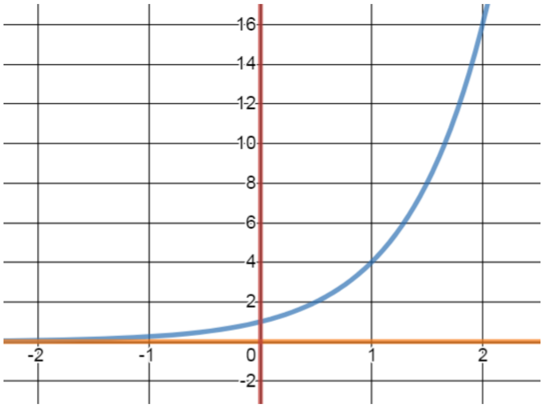
Second, we analyze what the y-values do when $x \to -\infty$. Consider the three ordered pairs (confirm with your calculator):

(–5, 0.000977) (–8, 0.000015) (–10, 0.000000954)

It appears that as x grows smaller (towards $-\infty$), the y-values are positive numbers shrinking toward zero. That is,

As $x \to -\infty$, $y \to 0$

The ordered pairs we listed and the limits we calculated lead us to the following graph:



The **domain** is ℝ. Also, there is no vertical **asymptote**, but the line $y = 0$ (the x-axis) is a **horizontal asymptote**.

Last, it appears that there is no ***x*-intercept** on our graph. Even more importantly, this confirms that

The equation $4^x = 0$ has <u>no</u> solution.

EXAMPLE 4: **Graph:** $y = \left(\frac{1}{2}\right)^x$

Solution: Let's get right to some ordered pairs.

If $x = -3$, then $y = \left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$, which gives us the ordered pair (–3, 8). It's now your job to verify each of the following ordered pairs in our function:

(0, 1) $(1, \frac{1}{2})$ $(2, \frac{1}{4})$ $(3, \frac{1}{8})$ $(4, \frac{1}{16})$

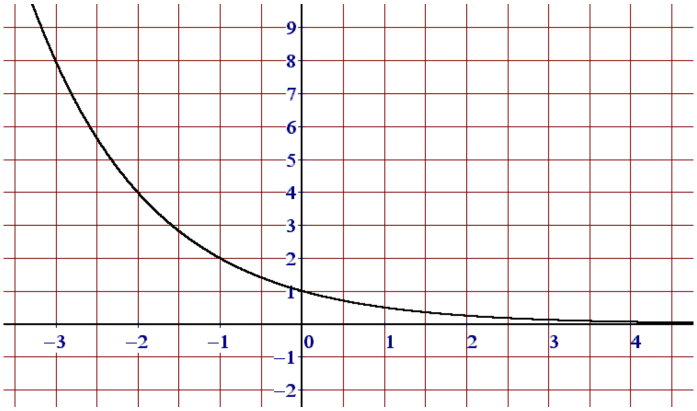
(–1, 2) (–2, 4) (–3, 8) (–4, 16)

The *y*-intercept is (0, 1). There is no *x*-intercept, since the equation $\left(\frac{1}{2}\right)^x = 0$ has no solution. The ordered pairs listed above give credence to the following limits:

As $x \to \infty$, $y \to 0$ and As $x \to -\infty$, $y \to \infty$

The ordered pairs and the limits lead us to the following graph:

We can see that the domain of this function is ℝ. Notice also that we have a horizontal asymptote at $y = 0$, but there is no vertical asymptote.



EXAMPLE 5: **Graph:** $f(x) = 2^{-x}$

Solution: Two observations and we'll be done in a jiffy. First, we know that $f(x)$ can be written simply as y. Second, check out the following calculation:

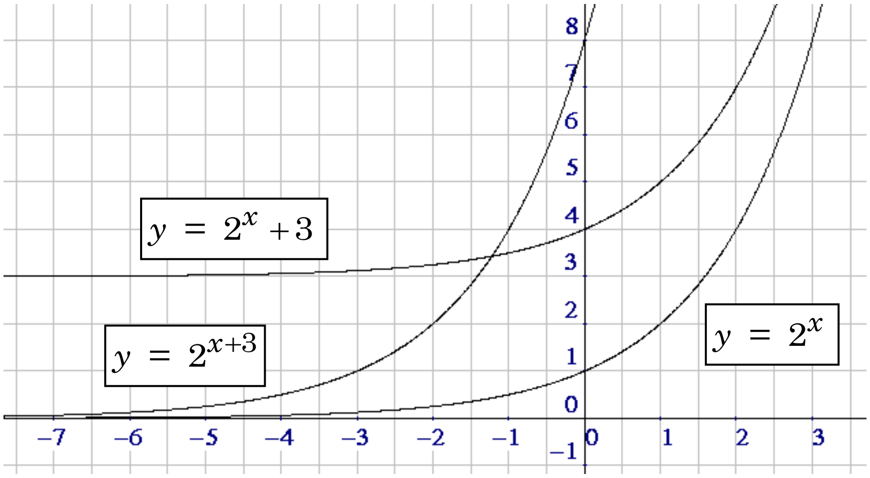
$$2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$$

In other words, the original formula can be written $y = \left(\frac{1}{2}\right)^x$, which we just finished graphing. So the solution to this problem is identical to that of the previous example.

EXAMPLE 6: **Graph:** $y = 2^x$ **and** $y = 2^x + 3$ **and** $y = 2^{x+3}$.

Solution: Let's make one table showing x and all the y-values at once and a single grid containing all the graphs at once.

x	2^x	$2^x + 3$	2^{x+3}
−6	1/64	3 1/64	1/8
−5	1/32	3 1/32	1/4
−4	1/16	3 1/16	1/2
−3	1/8	3 1/8	1
−2	1/4	3 1/4	2
−1	1/2	3 1/2	4
0	1	4	8
1	2	5	16
2	4	7	32
3	8	11	64



You should note that the graph of $2^x + 3$ is just the graph of 2^x but shifted 3 units up. Also, we see that the graph of 2^{x+3} is the result of taking the graph of 2^x and shifting it 3 units to the left.

Homework

4. Graph: $f(x) = 3^x$

5. Graph: $y = 3^x - 2$

6. Graph: $y = 3^{x+2}$

7. Graph: $g(x) = \left(\frac{1}{3}\right)^x$

8. Graph: $h(x) = 3^{-x}$

☐ *THE LEGAL BASES OF AN EXPONENTIAL FUNCTION*

In the previous section we graphed exponential functions with bases 4, $\frac{1}{2}$, 2, 3, and $\frac{1}{3}$. Now it's time to figure out exactly which bases we'll allow in the exponential function

$$f(x) = b^x$$

Whatever values of ***b*** we allow to be the base of an exponential function, we'd like the domain of the function (the legal x-values) to be

ℝ, the set of real numbers. And we don't want the exponential function to degenerate into some simple function that doesn't possess the "exponential" properties we've seen up until now.

$b < 0$ What about negative bases? Consider $f(x) = (-4)^x$. If we choose $x = \frac{1}{2}$, the functional value is $(-4)^{1/2} = \sqrt{-4}$, not a real number. We thus disallow any base b that is negative.

$b = 0$ Now consider $f(x) = 0^x$. But 0^x is fraught with problems. For example, if $x = 0$, we get 0^0. Can we assign a value to 0^0? On the one hand, 0 to any power should be 0. On the other hand, anything to the 0 power is supposed to be 1. So 0^0 is meaningless (but can be solved in Calculus). Even worse, consider 0^{-2}. Since a negative exponent indicates reciprocal, we get $\frac{1}{0^2} = \frac{1}{0}$, which is undefined. All in all, a base of 0 really stinks.

$0 < b < 1$ These bases are just fine. We used bases of $\frac{1}{2}$ and $\frac{1}{3}$ in the previous section. Even the number $\frac{1}{\pi}$ would be a legal base, although I've never seen it used.

$b = 1$ This gives us the function $f(x) = 1^x$, which is the function $f(x) = 1$, a constant function (the horizontal line $y = 1$). Exponential functions aren't supposed to be flat, so b shouldn't be allowed to be 1.

$b > 1$ Any base bigger than 1 is appropriate. In fact, in computer science a base of **2** is very popular. In basic science, the best base is **10** (for things like acids, earthquakes, and the volume of sound). And in calculus and the more advanced sciences, we use a strange number you may have seen: "***e***".

Homework

9. Describe precisely the legal bases for an exponential function.

10. Explain why –9 is not a good base for an exponential function.

11. Which of the following real numbers are legal bases for an exponential function?

$$-1 \qquad -.01 \qquad 0 \qquad \tfrac{2}{3} \qquad 0.987 \qquad 1 \qquad \pi \qquad 200$$

Review Problems

12. T/F: $y = x^3$ is an exponential function.
13. Describe the real numbers which can be used as the base of an exponential function.
14. Explain why 1 is not a good base for an exponential function.
15. Give a function which is both an exponential function <u>and</u> a polynomial function.
16. Graph $y = 5^x$.
17. Graph $y = 3^{-x} - 2$ and state its horizontal asymptote.
18. Let $f(x) = 2^x$. Now let g be the graph which results from taking the graph of f and shifting it 7 units to the left and 4 units up. Find a formula for g.
19. T/F: $y = 3^x$ is an exponential function.
20. T/F: In the exponential function $f(x) = b^x$, b can be any positive real number.

21. What is the domain of the function $y = 10^{7x+1} - 10$?

22. Find all the asymptotes of the function $f(x) = 99^x$.

23. Explain why the graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = 3^{-x}$ are the same.

24. How does the graph of $f(x) = 5^{x-2} + 4$ compare with that of $y = 5^x$?

25. T/F: All exponential functions are increasing functions.

26. Explain why 0 is not a good base for an exponential function.

27. True/False:

    a. $y = x^3$ is an exponential function.

    b. $y = \pi^x$ is an exponential function.

    c. The domain of the function $y = 4^x$ is $[0, \infty)$.

    d. The function $y = x^x$ is neither polynomial nor exponential.

    e. The function $y = 2^x - 1$ has an x-intercept.

    f. For the function above, as $x \to -\infty$, $y \to 0$.

    g. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$. As $x \to \infty$, $y \to \infty$.

    h. Compared to the graph of $y = 4^x$, the graph of $y = 4^{x-5}$ is five units lower.

    i. Any real number $b > 0$ is a legal base for an exponential function.

    j. Any real number $b \ge 0$, but not equal to 1, is a legal base for an exponential function.

    k. The number $\pi + \sqrt{2}$ is a legal base for an exponential function.

    l. All exponential functions are decreasing functions.

Solutions

1. a.

t	1	2	3	4	5	6	7	8	9	10
$9t$	9	18	27	36	45	54	63	72	81	90
3^t	3	9	27	81	243	729	2187	6561	19683	59049

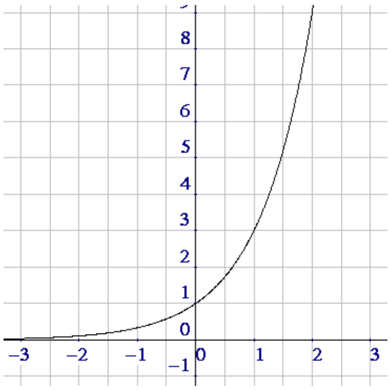
b. linear　　c. exponential　　d. first two hours

e. $t = 3$　　f. 59,049 / 90 $\approx$ 656

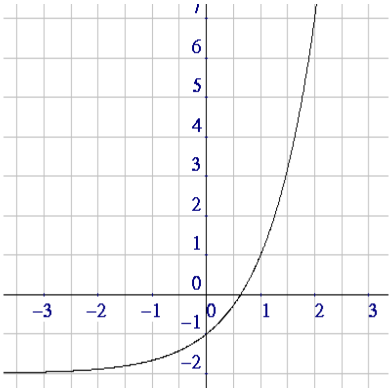
2. a. 64　　b. $\frac{1}{64}$　　c. $\frac{1}{2}$　　d. 8

3. x can be all kinds of numbers, so the domain is probably $\mathbb{R}$.

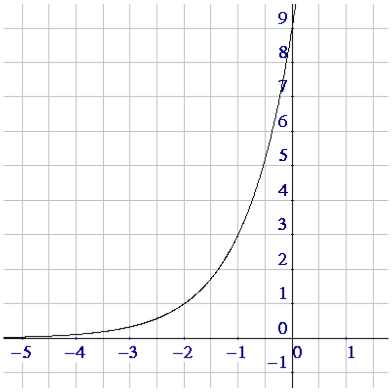
4.



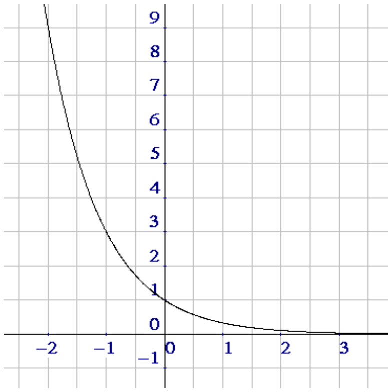
5.



6.



7.



8. Same graph as #14, since $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

9. The base must be positive but not equal to 1. That is, the function $f(x) = b^x$ is an exponential function if $b > 0$, but $b \neq 1$. In other words, the base b must be in the set: $(0, \infty) - \{1\}$.

10. If we consider the exponential function $y = (-9)^x$, then we could not use $x = \frac{1}{2}$, since $y = (-9)^{\frac{1}{2}} = \sqrt{-9} \notin \mathbb{R}$.

11. 2/3, 0.987, π, 200

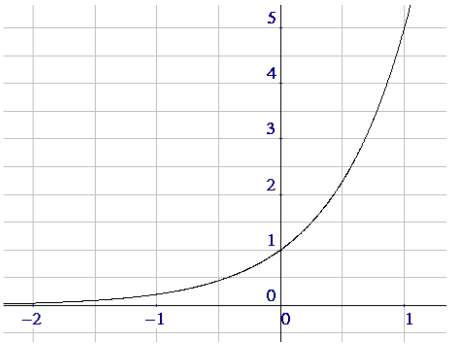
12. False

13. $\{x \in \mathbb{R} \mid x > 0, x \neq 1\}$ –OR– Any positive real number $\neq 1$
–OR– $(0, 1) \cup (1, \infty)$ –OR– $(0, \infty) - \{1\}$

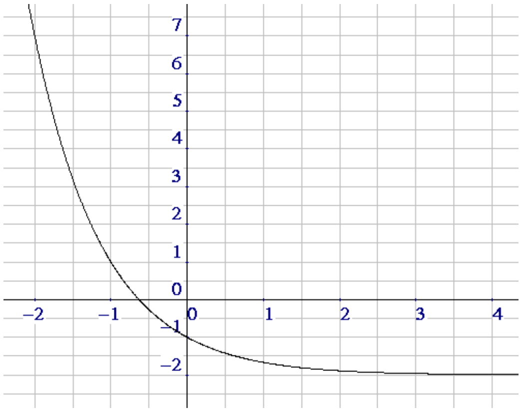
14. If b were 1, then $f(x) = b^x$ would become $f(x) = 1^x = 1$, which is a simple constant function whose graph is a horizontal line, rather useless to describe exponential growth and decay.

15. Ain't no such animal

16.



17.



Horizontal asymptote: $y = -2$

18. $g(x) = 2^{x+7} + 4$

19. T **20**. F (any positive real number ≠ 1)

21. $\mathbb{R}$ **22**. horiz: $y = 0$; vert: none

23. Because $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

24. The graph of f is the graph of y shifted 2 units to the right and 4 units up.

25. F

26. Because if $f(x) = 0^x$, then $f(x) = 0$, which is just a horizontal line. Even a better reason: If $f(x) = 0^x$ and we choose $x = -4$, we get an output of 0^{-4}, which is $1/0^4$, which is 1/0, which is verboten!

27. a. F b. T c. F d. T e. T f. F

g. F h. F i. F j. F k. T l. F

"The most beautiful experience
we can have is the mysterious.
It is the fundamental emotion
which stands at the cradle
of true art and true science."

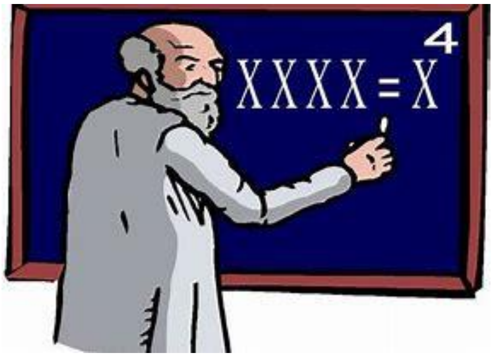
Albert Einstein

FRACTIONAL EXPONENTS

We know what n^5 means ($n \cdot n \cdot n \cdot n \cdot n$) , and we might know what x^{-3} means ($1/x^3$). You should also commit to memory that $w^1 = w$ for any number *w*, and that $y^0 = 1$ for any number *y* (that isn't zero). But what could something like $a^{5/3}$ mean . . . an exponent that's a fraction?? Stick around and you'll find out.

☐ *REVIEW OF EXPONENT LAWS*

Let's begin by reviewing The Five Laws of Exponents:



$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$(x\,y)^a = x^a\,y^a$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

Homework

1. Simplify each expression (assumes you've studied *Negative Exponents*):

a. x^3x^7

b. $\dfrac{a^{10}}{a^5}$

c. $(z^3)^5$

d. $\left(\dfrac{x}{y}\right)^9$

e. $(mn)^6$

f. r^4t^5

g. $a^2 + a^4$

h. $g^6 + g^6$

i. $(x + y)^2$

j. x^4x^{-3}

k. $(c^{-3})^{-5}$

l. $(ab)^{-4}$

m. $\dfrac{x^{-8}}{x^{-6}}$

n. $\dfrac{a^{-4}}{a^4}$

o. $\left(\dfrac{u}{w}\right)^{-3}$

p. $(a - b)^2$

☐ *THE MEANING OF A FRACTIONAL EXPONENT*

Now for a new kind of exponent: <u>fractional</u>. For example, is there any reasonable meaning for the expression "9 to the 1/2 power": $9^{1/2}$? To determine the meaning of this number, we can give $9^{1/2}$ a name; call it R. Now check out the following:

$$R = 9^{1/2}$$

$$\Rightarrow \quad R^{\mathbf{2}} = (9^{1/2})^{\mathbf{2}} \qquad \text{(square each side, just to see what happens)}$$

$$\Rightarrow \quad R^2 = 9^{\frac{1}{2}\cdot 2} \qquad \text{(law of exponents)}$$

$$\Rightarrow \quad R^2 = 9^1 \qquad \text{(1/2 and 2 are reciprocals)}$$

$$\Rightarrow \quad R^2 = 9$$

$$\Rightarrow \quad R = \pm 3 \qquad \text{(don't forget that there are 2 solutions)}$$

Since R is equal to both $9^{1/2}$ and ±3, it follows that

$$9^{1/2} = \pm 3$$

But we can't have two answers to one question, so we'll keep the positive 3 and ignore the negative 3, and declare that

$$9^{1/2} = 3$$

What are we saying here? Since we know that $\sqrt{9} = 3$, we're saying that

$9^{1/2}$ simply means the **positive <u>square root</u> of 9.**

Without any more gory details, we could prove that all fractional exponents translate to radicals:

$$\boxed{x^{1/2} = \sqrt{x} \qquad x^{1/3} = \sqrt[3]{x} \qquad x^{1/4} = \sqrt[4]{x}}$$

Examples:

$$9^{1/2} = 3 \qquad 100^{1/2} = 10 \qquad 25^{1/2} = 5$$

$$64^{1/3} = 4 \qquad 8^{1/3} = 2 \qquad 125^{1/3} = 5$$

$$16^{1/4} = 2 \qquad 81^{1/4} = 3 \qquad 256^{1/4} = 4$$

Summary: Assuming n is a natural number bigger than 1:

$$\boxed{x^{1/n} = \sqrt[n]{x}} \qquad n = 2, 3, 4, \ldots$$

EXAMPLE 1: **Evaluate each expression with a fractional exponent:**

A. $225^{1/2} = \sqrt{225} = \mathbf{15}$

B. $125^{1/3} = \sqrt[3]{125} = \mathbf{5}$

C. $81^{1/4} = \sqrt[4]{81} = \mathbf{3}$

D. $(-32)^{1/5} = \sqrt[5]{-32} = \mathbf{-2}$

E. $-16^{1/2} = -\sqrt{16} = \mathbf{-4}$

F. $(-16)^{1/2} = \sqrt{-16} =$ **Not a real number**

G. $-81^{1/4} = -\sqrt[4]{81} = \mathbf{-3}$

H. $(-81)^{1/4} = \sqrt[4]{-81} =$ **Not a real number**

Homework

2. Explain why $25^{1/2}$ is a real number, but $(-25)^{1/2}$ is not.

3. Evaluate each expression:

a. $36^{1/2}$ b. $8^{1/3}$ c. $16^{1/4}$ d. $32^{1/5}$

e. $625^{1/2}$ f. $1^{1/3}$ g. $0^{1/4}$ h. $243^{1/5}$

i. $-25^{1/2}$ j. $-49^{1/2}$ k. $(-64)^{1/2}$ l. $(-16)^{1/4}$

m. $(-64)^{1/3}$ n. $(-1)^{1/5}$ o. $(-32)^{1/5}$ p. $(-1)^{1/4}$

4. Convert each expression to radical form:

a. $x^{1/2}$ b. $y^{1/3}$ c. $z^{1/4}$ d. $w^{1/5}$

e. $(ab)^{1/2}$
f. $ab^{1/2}$
g. $xy^{1/3}$
h. $(xy)^{1/3}$
i. $y+z^{1/2}$
j. $(y+z)^{1/2}$
k. $(a-b)^{1/3}$
l. $(Q+R-T)^{1/4}$

5. Convert each expression to exponent form:

a. $\sqrt{x}$
b. $\sqrt[4]{y}$
c. $\sqrt[3]{z}$
d. $\sqrt[5]{n}$
e. $a\sqrt{b}$
f. $\sqrt{ab}$
g. $x\sqrt[4]{y}$
h. $\sqrt[3]{tw}$
i. $\sqrt{x+y}$
j. $\sqrt[3]{p-q}$
k. $\sqrt[4]{a+n}$
l. $\sqrt[6]{x-x}$

6. True/False:

a. The expression $x^{1/2}$ is always defined.

b. The expression $x^{1/3}$ is always defined.

❑ MORE FRACTIONAL EXPONENTS

The previous problems each had a numerator of 1 in the fractional exponent. What about an expression like $27^{2/3}$? What could this mean? Let's dissect $27^{2/3}$ using our laws of exponents to determine the value of this number.

$$\begin{aligned}
& 27^{2/3} && \text{(the power of 27 we're analyzing)} \\
= \quad & 27^{\frac{1}{3}\cdot 2} && \text{(certainly } \tfrac{1}{3}\cdot 2 \ = \ \tfrac{2}{3}\text{)} \\
= \quad & \left(27^{1/3}\right)^2 && \text{(law of exponents: } (x^a)^b \ = \ x^{ab}\text{)} \\
= \quad & \left(\sqrt[3]{27}\right)^2 && \text{(a 1/3 exponent indicates cube root)} \\
= \quad & 3^2 && \text{(the cube root of 27 is 3)} \\
= \quad & \mathbf{9} && \text{(3 squared is 9)}
\end{aligned}$$

Here's another example. Let's calculate $16^{5/4}$:

$$\mathbf{16^{5/4}} = 16^{\frac{1}{4}\cdot 5} = \left(16^{1/4}\right)^5 = \left(\sqrt[4]{16}\right)^5 = 2^5 = \mathbf{32}$$

And a third example:

$$\mathbf{243^{2/5}} = 243^{\frac{1}{5}\cdot 2} = \left(243^{1/5}\right)^2 = \left(\sqrt[5]{243}\right)^2 = 3^2 = \mathbf{9}$$

Homework

7. Using the three examples above as a guide, find the value of each fractional power:

a. $8^{2/3}$ b. $4^{3/2}$ c. $9^{3/2}$ d. $16^{5/2}$

e. $27^{4/3}$ f. $27^{2/3}$ g. $16^{3/4}$ h. $32^{7/5}$

What's really going on here? There must be a simpler way to view a fractional exponent, and thus a simpler way to calculate one. If you look at the above examples and homework you just completed, you may have noticed that the denominator of the fractional exponent indicated a root, while the numerator indicated a power. For example, in the first example above, we showed that $27^{2/3}$ was calculated by first taking the cube root of 27, and then squaring that result to get **9**.

Thus, a problem like $16^{5/2}$ is calculated quickly as the square root of 16, raised to the fifth power, which is 4^5, which is **1,024**.

In summary,

$$\boxed{x^{p/q} = \left(\sqrt[q]{x}\right)^{p}}$$

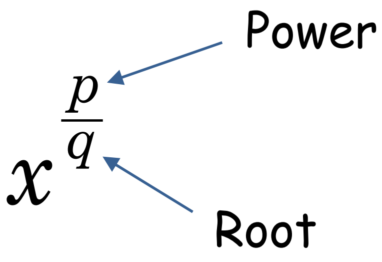
EXAMPLE:

$$8^{5/3}$$

$$= \left(\sqrt[3]{8}\right)^{5}$$

$$= 2^{5}$$

$$= 32$$



One more example for this section: To calculate $\mathbf{64^{2/3}}$, think cube root of 64, raised to the second power, which is 4^2, which is **16**.

Homework

8. Using the power-root idea, find the value of each fractional power:

a. $8^{4/3}$

b. $4^{1/2}$

c. $9^{5/2}$

d. $16^{3/2}$

e. $27^{2/3}$

f. $27^{4/3}$

g. $16^{5/4}$

h. $32^{6/5}$

i. $8^{2/3}$

j. $4^{3/2}$

k. $9^{3/2}$

l. $16^{5/2}$

m. $27^{4/3}$

n. $125^{2/3}$

o. $16^{3/4}$

p. $32^{7/5}$

❑ NEGATIVE FRACTIONAL EXPONENTS

Since a negative exponent indicates a reciprocal, we can combine negative exponents (assuming you've studied them) with the fractional exponents we're learning now. Recalling that $x^{-n} = \dfrac{1}{x^n}$, we can work the following examples.

EXAMPLE 2: **Evaluate each expression:**

A. $9^{-1/2} = \dfrac{1}{9^{1/2}} = \dfrac{1}{\sqrt{9}} = \mathbf{\dfrac{1}{3}}$

B. $-27^{-1/3} = -\dfrac{1}{27^{1/3}} = -\dfrac{1}{\sqrt[3]{27}} = -\mathbf{\dfrac{1}{3}}$

C. $8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{\left(\sqrt[3]{8}\right)^2} = \dfrac{1}{2^2} = \mathbf{\dfrac{1}{4}}$

D. $9^{-5/2} = \dfrac{1}{9^{5/2}} = \dfrac{1}{\left(\sqrt{9}\right)^5} = \dfrac{1}{3^5} = \mathbf{\dfrac{1}{243}}$

E. $-16^{-3/4} = -\dfrac{1}{16^{3/4}} = -\dfrac{1}{\left(\sqrt[4]{16}\right)^3} = -\dfrac{1}{2^3} = -\mathbf{\dfrac{1}{8}}$

F. $(-16)^{-3/2} = \dfrac{1}{(-16)^{3/2}} = \dfrac{1}{\left(\sqrt{-16}\right)^3} =$ **Not a real number**

G. $(-8)^{-2/3} = \dfrac{1}{(-8)^{2/3}} = \dfrac{1}{\left(\sqrt[3]{-8}\right)^2} = \dfrac{1}{(-2)^2} = \mathbf{\dfrac{1}{4}}$

Homework

9. Evaluate each expression:

a. $9^{-3/2}$ b. $27^{-2/3}$ c. $8^{-4/3}$ d. $8^{-1/3}$

e. $16^{-5/4}$ f. $81^{-1/4}$ g. $81^{-3/4}$ h. $32^{-1/5}$

i. $-8^{-4/3}$ j. $(-25)^{-3/2}$ k. $(-64)^{-2/3}$ l. $(-32)^{-6/5}$

10. Convert each expression to radical form:

a. $x^{2/3}$ b. $y^{5/4}$ c. $z^{1/7}$

d. $(a+b)^{3/2}$ e. $(x-y)^{4/3}$ f. $(xy)^{4/5}$

g. $uw^{2/3}$ h. $x+y^{4/5}$ i. $(ab+c)^{5/6}$

11. Convert each expression to exponent form:

a. $\sqrt[3]{t}$ b. $\sqrt[5]{z}$ c. $\sqrt[4]{ab}$

d. $x\sqrt[3]{y}$ e. $\left(\sqrt{a}\right)^3$ f. $\left(\sqrt[3]{p}\right)^2$

g. $\left(\sqrt[4]{w}\right)^5$ h. $\left(\sqrt[7]{a+b}\right)^3$ i. $\left(\sqrt{x-y}\right)^{10}$

❑ *THE LAWS OF EXPONENTS*

The same five laws of exponents we've used with all the previous exponents still work just fine with fractional exponents.

EXAMPLE 3: **Simplify each expression:**

A. $a^{1/2}a^{1/3} = a^{1/2+1/3} = a^{3/6+2/6} = \boldsymbol{a^{5/6}}$

(add the exponents)

B. $y^{1/2}y^{1/2} = y^{1/2+1/2} = y^1 = \boldsymbol{y}$

(add the exponents)

C. $\left(x^{3/4}\right)^{7/2} = x^{(3/4)(7/2)} = \boldsymbol{x^{21/8}}$

(multiply the exponents)

D. $\dfrac{n^{5/6}}{n^{2/3}} = n^{5/6-2/3} = n^{5/6-4/6} = \boldsymbol{n^{1/6}}$

(subtract the exponents)

E. $(ab)^{4/5} = \boldsymbol{a^{4/5}b^{4/5}}$

(raise each factor to the 4/5 power)

F. $\left(\dfrac{a}{b}\right)^{5/3} = \boldsymbol{\dfrac{a^{5/3}}{b^{5/3}}}$

(raise top and bottom to the 5/3 power)

G. $\dfrac{w^{1/2}}{w^{4/5}} = w^{1/2-4/5} = w^{5/10-8/10} = w^{-3/10} = \boldsymbol{\dfrac{1}{w^{3/10}}}$

(subtract the exponents) (LCD) (a negative exponent means reciprocal)

Homework

12. Simplify each expression:

a. $x^{1/2}x^{2/3}$

b. $\dfrac{y^{4/5}}{y^{1/5}}$

c. $\dfrac{w^{1/2}}{w^{2/3}}$

d. $\left(a^{2/7}\right)^7$

e. $\left(\dfrac{p}{q}\right)^{3/8}$

f. $t^{4/5}t^{1/3}$

g. $\left(k^{5/2}\right)^{2/5}$

h. $(wz)^{2/3}$

EXAMPLE 4: **Simplify each expression:**

A. $x^{-1/2}x^{-2/3} = x^{-1/2\,-\,2/3} = x^{-3/6\,-\,4/6} = x^{-7/6} = \mathbf{\dfrac{1}{x^{7/6}}}$

B. $\dfrac{n^{1/2}}{n^{-4/5}} = n^{1/2\,-\,(-4/5)} = n^{1/2\,+\,4/5} = n^{5/10\,+\,8/10} = \mathbf{n^{13/10}}$

C. $\left(x^{2/3}\right)^{-1/4} = x^{(2/3)(-1/4)} = x^{-1/6} = \mathbf{\dfrac{1}{x^{1/6}}}$

D. $(xy)^{-2/3} = x^{-2/3}y^{-2/3} = \dfrac{1}{x^{2/3}}\cdot\dfrac{1}{y^{2/3}} = \mathbf{\dfrac{1}{x^{2/3}y^{2/3}}}$

E. $\left(\dfrac{g}{h}\right)^{-4/3} = \dfrac{g^{-4/3}}{h^{-4/3}} = \dfrac{\dfrac{1}{g^{4/3}}}{\dfrac{1}{h^{4/3}}} = \dfrac{1}{g^{4/3}}\times\dfrac{h^{4/3}}{1} = \mathbf{\dfrac{h^{4/3}}{g^{4/3}}}$

Homework

13. Simplify each expression:

a. $x^{4/5}x^{-3/5}$

b. $y^{1/3}y^{-5/3}$

c. $\left(a^{-1/2}\right)^{-2/3}$

d. $(abc)^{-3/4}$

e. $\left(\dfrac{w}{z}\right)^{-2/5}$

f. $\dfrac{n^{-1/2}}{n^{-2/3}}$

g. $\dfrac{a^{-3}}{a^{5/2}}$

h. $\dfrac{x}{x^{-2/3}}$

i. $\left(\left(c^{1/2}\right)^{-4/3}\right)^{-3/2}$

Review Problems

14. Convert $\sqrt[4]{x+y}$ to exponent form.

15. Convert $ab^{2/3}$ to radical form.

16. Convert $(a+b)^{5/2}$ to radical form.

17. Convert $\sqrt{a^3-x^3}$ to exponent form.

18. Evaluate: a. $9^{1/2}$ b. $64^{1/3}$ c. $81^{1/4}$ d. $32^{1/5}$

19. Evaluate: a. $8^{2/3}$ b. $27^{4/3}$ c. $32^{2/5}$ d. $16^{3/4}$

20. Evaluate: a. $-9^{1/2}$ b. $(-9)^{1/2}$ c. $(-8)^{1/3}$ d. $(-16)^{1/4}$

21. Evaluate: a. $9^{-3/2}$ b. $8^{-4/3}$ c. $125^{-4/3}$ d. $-81^{-1/4}$

22. Simplify: a. $x^{1/2}x^{4/5}$ b. $\dfrac{a^{1/3}}{a^{2/5}}$ c. $(ab)^{4/7}$

23. Simplify: a. $\left(\dfrac{a}{b}\right)^{2/7}$ b. $\left(x^{2/3}\right)^{3/5}$ c. $a^{1/2}a^{1/3}a^{1/4}$

24. Simplify: a. $y^{1/2}y^{-1/2}$ b. $\dfrac{n^{1/3}}{n^{-4/3}}$ c. $(PQ)^{-2/3}$

25. Simplify: a. $\left(\dfrac{x}{w}\right)^{-7/10}$ b. $\left(p^{-2/3}\right)^{5/6}$ c. $x^{2/3}+x^{1/3}$

Solutions

1. a. x^{10} b. a^5 c. z^{15} d. $\dfrac{x^9}{y^9}$ e. m^6n^6

f. As is g. As is h. $2g^6$ i. $x^2 + 2xy + y^2$

j. x k. c^{15} l. $\frac{1}{a^4b^4}$ m. $\frac{1}{x^2}$ n. $\frac{1}{a^8}$

o. $\frac{w^3}{u^3}$ p. Assuming you've learned how to multiply binomials, the answer is $a^2 - 2ab + b^2$

2. If w were 0, we'd have $0^{-4} = \frac{1}{0^4} = \frac{1}{0} =$ Undefined

3. a. $25^{1/2} = (5^2)^{1/2} = 5^{2 \cdot \frac{1}{2}} = 5^1 = 5$ b. same idea; result is 9

c. same idea; result is 12 d. same idea; result is 7

e. $8^{1/3} = (2^3)^{1/3} = 2^{3 \cdot \frac{1}{3}} = 2^1 = 2$ f. same idea; result is 5

g. same idea; result is 3 h. same idea; result is 6

i. $81^{1/4} = (3^4)^{1/4} = 3^{4 \cdot \frac{1}{4}} = 3^1 = 3$ j. same idea; result is 4

4. $25^{1/2} = \sqrt{25} = 5$, a real number. But $(-25)^{1/2} = \sqrt{-25}$, not a real number.

5. a. 6 b. 2 c. 2 d. 2 e. 25 f. 1 g. 0 h. 3 i. –5
j. –7 k. Not real l. Not real m. –4 n. –1 o. –2
p. Not real

6. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$ d. $\sqrt[5]{w}$ e. $\sqrt{ab}$
f. $a\sqrt{b}$ g. $x\sqrt[3]{y}$ h. $\sqrt[3]{xy}$ i. $y+\sqrt{z}$ j. $\sqrt{y+z}$
k. $\sqrt[3]{a-b}$ l. $\sqrt[4]{Q+R-T}$

7. a. $x^{1/2}$ b. $y^{1/4}$ c. $z^{1/3}$ d. $n^{1/5}$ e. $ab^{1/2}$ f. $(ab)^{1/2}$
g. $xy^{1/4}$ h. $(tw)^{1/3}$ i. $(x+y)^{1/2}$ j. $(p-q)^{1/3}$ k. $(a+n)^{1/4}$ l. 0

8. a. False; if $x = -9$, for instance, then $(-9)^{1/2} = \sqrt{-9}$ which is not a real number, which means that $x^{1/2}$ is undefined in this class when $x = -9$. In fact, $x^{1/2}$ is undefined whenever x is a negative number.

b. True; since $x^{1/3} = \sqrt[3]{x}$, and since the cube root is defined whether x is positive, zero, or negative, $x^{1/3}$ is always defined.

9. a. $8^{2/3} = 8^{\frac{1}{3}\cdot 2} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

b. 8 c. 27 d. 1024 e. 81 f. 9 g. 8

h. $32^{7/5} = 32^{\frac{1}{5}\cdot 7} = \left(32^{1/5}\right)^7 = \left(\sqrt[5]{32}\right)^7 = 2^7 = 128$

10. a. $8^{4/3}$ is the cube root of 8, raised to the 4th power: $2^4 = 16$
b. 2
c. $9^{5/2}$ is the square root of 9, raised to the 5th power: $3^5 = 243$
d. 64 e. 9 f. 81 g. 32
h. $32^{6/5}$ is the fifth root of 32, raised to the 6th power: $2^6 = 64$
i. 4 j. 8 k. 27 l. 1024 m. 81 n. 25 o. 8 p. 128

11. a. $\frac{1}{27}$ b. $\frac{1}{9}$ c. $\frac{1}{16}$ d. $\frac{1}{2}$ e. $\frac{1}{32}$ f. $\frac{1}{3}$
g. $\frac{1}{27}$ h. $\frac{1}{2}$ i. $-\frac{1}{16}$ j. Not real k. $\frac{1}{16}$ l. $\frac{1}{64}$

12. a. $\left(\sqrt[3]{x}\right)^2$ b. $\left(\sqrt[4]{y}\right)^5$ c. $\sqrt[7]{z}$ d. $\left(\sqrt{a+b}\right)^3$ e. $\left(\sqrt[3]{x-y}\right)^4$
f. $\left(\sqrt[5]{xy}\right)^4$ g. $u\left(\sqrt[3]{w}\right)^2$ h. $x+\sqrt[5]{y^4}$ i. $\left(\sqrt[6]{ab+c}\right)^5$

13. a. $t^{1/3}$ b. $z^{1/5}$ c. $(ab)^{1/4}$ d. $xy^{1/3}$ e. $a^{3/2}$
f. $p^{2/3}$ g. $w^{5/4}$ h. $(a+b)^{3/7}$ i. $(x-y)^{10/2} = (x-y)^5$

14. a. $x^{7/6}$ b. $y^{3/5}$ c. $\frac{1}{w^{1/6}}$ d. a^2

e. $\dfrac{p^{3/8}}{q^{3/8}}$ f. $t^{17/15}$ g. k h. $w^{2/3}z^{2/3}$

15. a. $x^{1/5}$ b. $\dfrac{1}{y^{4/3}}$ c. $a^{1/3}$ d. $\dfrac{1}{a^{3/4}b^{3/4}c^{3/4}}$

e. $\dfrac{z^{2/5}}{w^{2/5}}$ f. $n^{1/6}$ g. $\dfrac{1}{a^{11/2}}$ h. $x^{5/3}$

i. c

16. $(x+y)^{1/4}$ **17**. $a\left(\sqrt[3]{b}\right)^2$

18. $\left(\sqrt{a+b}\right)^5$ or $\sqrt{(a+b)^5}$

19. $(a^3-x^3)^{1/2}$

20. a. 3 b. 4 c. 3 d. 2

21. a. 4 b. 81 c. 4 d. 8

22. a. –3 b. Not real c. –2 d. Not real

23. a. $\dfrac{1}{27}$ b. $\dfrac{1}{16}$ c. $\dfrac{1}{625}$ d. $-\dfrac{1}{3}$

24. a. $x^{13/10}$ b. $\dfrac{1}{a^{1/15}}$ c. $a^{4/7}b^{4/7}$

25. a. $\dfrac{a^{2/7}}{b^{2/7}}$ b. $x^{2/5}$ c. $a^{13/12}$

26. a. 1 b. $n^{5/3}$ c. $\dfrac{1}{P^{2/3}Q^{2/3}}$

27. a. $\dfrac{w^{7/10}}{x^{7/10}}$ b. $\dfrac{1}{p^{5/9}}$ c. As is

"If you limit your choices only to what seems possible or reasonable, you disconnect yourself from what you truly want, and all that is left is a compromise."

- Robert Fritz

CH XX – NEGATIVE EXPONENTS

The mass of a proton (the positively charged particle in the nucleus of every atom in the universe) is written in *scientific notation* as

$$\mathbf{1.67 \times 10^{-27}\ kilograms}$$

How do we interpret that? Is that a really big or a really small number? In this chapter, we will learn exactly what a negative exponent means.

❑ REVIEW OF THE FIVE LAWS OF EXPONENTS

Before we begin the discussion of negative exponents, it would be beneficial for us to review the meaning of an exponent and the Five Laws of exponents:

If n is a natural number ($n = 1, 2, 3, \ldots$), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

Now we recall the Five Laws of Exponents. The order doesn't make any difference, but I personally refer to the first one as the First Law of Exponents.

$$\boxed{\begin{array}{ccc} x^a x^b = x^{a+b} & \dfrac{x^a}{x^b} = x^{a-b} & (x^a)^b = x^{ab} \\[2ex] (xy)^a = x^a y^a & & \left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a} \end{array}}$$

❑ DEVELOPING THE MEANING OF A NEGATIVE EXPONENT

Consider the following power of x:

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot \mathbf{x^3} = x^{-3+3} = x^0 = 1$$

which implies that

$$x^{-3} \cdot x^3 = 1$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3:

$$\frac{x^{-3} \cdot x^3}{x^3} = \frac{1}{x^3}$$

$$\Rightarrow \quad \boxed{x^{-3} = \frac{1}{x^3}}$$

We can do the same thing in general. Consider x^{-n}, where n is any number at all:

$$x^{-n} \cdot x^n = x^{-n+n} = x^0 = 1 \qquad \Rightarrow \qquad x^{-n} = \frac{1}{x^n}$$

In short,

$$\boxed{x^{-n} = \frac{1}{x^n}}$$

for ANY number n, assuming $x \neq 0$.

This should convince you that a negative exponent means **reciprocal**. In fact, the exponent doesn't have to be a negative whole number. [Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals; in fact, $w^{-5/4} = \frac{1}{w^{5/4}}$ and $h^{-\sqrt{2\pi}} = \frac{1}{h^{\sqrt{2\pi}}}$.]

This fact, together with our knowledge of fractions and the Laws of Exponents, is all we need to understand the examples in this chapter.

EXAMPLE 1:

A. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

C. $-3^{-4} = -\frac{1}{3^4} = -\frac{1}{81}$

See the difference?

D. $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$

E. $\frac{1}{2^{-5}} = \frac{1}{\frac{1}{2^5}} = \frac{1}{\frac{1}{32}} = \frac{1}{1} \cdot \frac{32}{1} = 32$

F. $\dfrac{1}{10^{-3}} = \dfrac{1}{\dfrac{1}{10^3}} = \dfrac{1}{1}\cdot\dfrac{10^3}{1} = 10^3 = 1000$

G. $\left(\dfrac{2}{3}\right)^{-3} = \dfrac{1}{\left(\dfrac{2}{3}\right)^3} = \dfrac{1}{\dfrac{8}{27}} = \dfrac{1}{1}\cdot\dfrac{27}{8} = \dfrac{27}{8}$

H. $0^{-5} = \dfrac{1}{0^5} = \dfrac{1}{0} = \textbf{Undefined}$

Homework

1. Evaluate each expression:

a. 3^{-2} b. 2^{-5} c. 10^{-3} d. 5^{-4} e. 1^{-10}

f. -2^{-3} g. $(-3)^{-3}$ h. -10^{-4} i. $(-5)^{-1}$ j. -6^{-2}

k. $\dfrac{1}{3^{-2}}$ l. $\dfrac{1}{10^{-4}}$ m. $\dfrac{1}{5^{-3}}$ n. $\left(\dfrac{3}{4}\right)^{-2}$ o. $\left(\dfrac{1}{6}\right)^{-3}$

2. Explain why $x = 0$ is NOT allowed in the expression x^{-8}.

EXAMPLE 2:

A. $n^{-1} = \dfrac{1}{n}$

B. $a^{-12} = \dfrac{1}{a^{12}}$

C. $\dfrac{x}{y^{-3}} = \dfrac{x}{\dfrac{1}{y^3}} = x \cdot \dfrac{y^3}{1} = xy^3$

D. $\dfrac{x^{-7}}{a^{12}} = \dfrac{\dfrac{1}{x^7}}{a^{12}} = \dfrac{1}{x^7} \cdot \dfrac{1}{a^{12}} = \dfrac{1}{a^{12}x^7}$

E. $a^2b^{-3} = a^2 \cdot \dfrac{1}{b^3} = \dfrac{a^2}{b^3}$

F. $R^{-1}T^{-7} = \dfrac{1}{R} \cdot \dfrac{1}{T^7} = \dfrac{1}{RT^7}$

EXAMPLE 3:

A. $x + y^{-5} = x + \dfrac{1}{y^5} = \underbrace{\left[\dfrac{y^5}{y^5}\right]\dfrac{x}{1} + \dfrac{1}{y^5}}_{\text{to make a common denominator}} = \dfrac{xy^5}{y^5} + \dfrac{1}{y^5} = \dfrac{xy^5 + 1}{y^5}$

B. $a^{-2} + b^{-7} = \dfrac{1}{a^2} + \dfrac{1}{b^7} = \dfrac{1}{a^2}\left[\dfrac{b^7}{b^7}\right] + \dfrac{1}{b^7}\left[\dfrac{a^2}{a^2}\right] = \dfrac{b^7 + a^2}{a^2b^7}$

Homework

3. Simplify each expression (no negative exponents in the answer):

a. x^{-9}

b. a^{-30}

c. $\dfrac{a}{b^{-2}}$

d. $\dfrac{x^{-4}}{y^2}$

e. $\dfrac{w^3}{x^{-3}}$

f. $\dfrac{m^{-3}}{n^{-4}}$

g. $a^{-4}b^{-5}$

h. k^3p^{-5}

i. $h^{-5}z^{7}$ j. $\dfrac{w^{-5}}{w^{-5}}$ k. $x+y^{-1}$ l. $a^{-3}+b^{2}$

m. $p^{-1}+r^{-1}$ n. $w^{-3}-z^{-2}$ o. $a^{-3}-a^{-3}$ p. $a^{-1}b^{-2}c^{-3}$

EXAMPLE 4: Using the Five Laws of Exponents:

A. $x^{-6}x^{-8} = x^{-14} = \dfrac{\mathbf{1}}{\boldsymbol{x^{14}}}$

Add the exponents

B. $\left(N^{-5}\right)^{-7} = \boldsymbol{N^{35}}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \dfrac{1}{a^4}\cdot\dfrac{1}{b^4}\cdot\dfrac{1}{c^4} = \dfrac{\mathbf{1}}{\boldsymbol{a^4b^4c^4}}$

Apply exponent to
each factor

D. $\dfrac{w^5}{w^{15}} = w^{5-15} = w^{-10} = \dfrac{\mathbf{1}}{\boldsymbol{w^{10}}}$

Subtract the exponents

E. $\dfrac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \dfrac{\mathbf{1}}{\boldsymbol{x^5}}$

Subtract the exponents

F. $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{a^{-3}}{b^{-3}} = \dfrac{\dfrac{1}{a^3}}{\dfrac{1}{b^3}} = \dfrac{1}{a^3}\cdot\dfrac{b^3}{1} = \dfrac{\boldsymbol{b^3}}{\boldsymbol{a^3}}$

Apply exponent to
top and bottom

Another way: $\left(\frac{a}{b}\right)^{-3} = \frac{1}{\left(\frac{a}{b}\right)^3} = \frac{1}{\frac{a^3}{b^3}} = \frac{b^3}{a^3}$

G. $(x+y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of <u>two</u> terms — it is not a product — so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x+y)^{-2} = \frac{1}{(x+y)^2} = \frac{1}{(x+y)(x+y)} = \mathbf{\frac{1}{x^2+2xy+y^2}}$$

Homework

---

4. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^7$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^3$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x+y)^{-1}$ l. $(a-b)^{-2}$

m. $\frac{x^{-5}}{x^3}$ n. $\frac{z^5}{z^{-4}}$ o. $\frac{a^{-5}}{a^{-8}}$ p. $\frac{y^{-12}}{y^{-7}}$

q. $\left(\frac{a}{b}\right)^{-4}$ r. $\left(\frac{y}{x}\right)^{-1}$ s. $\left(\frac{a}{b+c}\right)^{-2}$ t. $\left(\frac{a+b}{xy+wz}\right)^{0}$

EXAMPLE 5: **Simplify each expression:**

A. $$(2x^{-8})(-5x^{6}) = (2)(-5)x^{-8}x^{6} = -10x^{-2} = -10\left(\frac{1}{x^{2}}\right) = -\frac{\mathbf{10}}{\boldsymbol{x}^{\mathbf{2}}}$$

B. $$\left(x^{2}y^{-3}\right)^{4} = \left(x^{2}\right)^{4}\left(y^{-3}\right)^{4} = x^{8}y^{-12} = x^{8}\left(\frac{1}{y^{12}}\right) = \frac{\boldsymbol{x}^{\mathbf{8}}}{\boldsymbol{y}^{\mathbf{12}}}$$

C. $$\left(\frac{a^{-3}}{b^{4}}\right)^{-5} = \frac{\left(a^{-3}\right)^{-5}}{\left(b^{4}\right)^{-5}} = \frac{a^{15}}{b^{-20}} = \frac{a^{15}}{\frac{1}{b^{20}}} = \frac{a^{15}}{1}\cdot\frac{b^{20}}{1} = \boldsymbol{a}^{\mathbf{15}}\boldsymbol{b}^{\mathbf{20}}$$

Final Note: In the Introduction, we learned about the mass of a proton. We're now in a position to see just what that number is:

$$1.67\times\mathbf{10^{-27}} = 1.67\times\frac{\mathbf{1}}{\mathbf{10^{27}}} = \frac{1.67}{1}\times\frac{1}{10^{27}} = \frac{1.67}{10^{27}}$$

$$= \frac{1.67}{1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$

$$= \boxed{0.00000000000000000000000000167 \text{ kilograms}}$$

Shortcut: Move the decimal point 27 places to the left.

[If available, check out the chapter on *Scientific Notation* for all the details and shortcuts regarding numbers like 1.67×10^{-27}.]

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

5. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^{7})$ b. $(-3y^{-3})(2y^{-5})$ c. $(-4a^{3})(a^{-7})$

d. $\left(u^{3}u^{-5}\right)^{-2}$ e. $\left(w^{-5}w^{-1}\right)^{7}$ f. $\left(a^{-2}b^{-3}\right)^{-5}$

g. $\left(t^{3}u^{-4}\right)^{3}$ h. $\left(\dfrac{x^{2}}{x^{5}}\right)^{-2}$ i. $\left(\dfrac{y^{-3}}{y^{-4}}\right)^{5}$

j. $\left(\dfrac{a^{-2}}{b^{5}}\right)^{-3}$ k. $\left(\dfrac{c^{3}}{d^{-4}}\right)^{-5}$ l. $\left(\dfrac{a^{3}b^{-4}}{x^{-12}z^{-2}}\right)^{0}$

6. Express each scientific notation number as a regular number:

a. 2.3×10^{7} b. 7.11×10^{-5} c. 5.09×10^{-10}

Review Problems

7. Evaluate: a. -2^{-4} b. $(-2)^{-4}$

8. Evaluate: a. $\left(\dfrac{4}{5}\right)^{-3}$ b. $\left(\dfrac{1}{9}\right)^{-1}$

9. Simplify: a. $\dfrac{a^{-2}}{b^{-3}}$ b. $\dfrac{x^{-9}}{x^{9}}$

10. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$

11. Simplify: a. $(x^{-10})^5$ b. $(z^{-5})^{-5}$

12. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^3y^{10})^{-2}$

13. Simplify: a. $\left(\dfrac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\dfrac{a^3}{b^{-4}}\right)^5$

14. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$

15. Explain why $x = 0$ is NOT allowed in the expression x^{-2}.

16. Express as a regular number: 4.9×10^{-13}

Solutions

1. a. $\frac{1}{9}$ b. $\frac{1}{32}$ c. $\frac{1}{1000}$ d. $\frac{1}{625}$ e. 1 f. $-\frac{1}{8}$

g. $-\frac{1}{27}$ h. $-\frac{1}{10{,}000}$ i. $-\frac{1}{5}$ j. $-\frac{1}{36}$ k. 9 l. 10,000

m. 125 n. $\frac{16}{9}$ o. 216

2. If x were 0 in the expression x^{-8}, we would have

$$x^{-8} = 0^{-8} = \frac{1}{0^8} = \frac{1}{0},$$ which is Undefined.

3. a. $\frac{1}{x^9}$ b. $\frac{1}{a^{30}}$ c. ab^2 d. $\frac{1}{x^4y^2}$ e. x^3w^3 f. $\frac{n^4}{m^3}$

g. $\frac{1}{a^4b^5}$ h. $\frac{k^3}{p^5}$ i. $\frac{z^7}{h^5}$ j. 1 k. $\frac{xy+1}{y}$ l. $\frac{1+a^3b^2}{a^3}$

m. $\frac{r+p}{pr}$ n. $\frac{z^2-w^3}{w^3z^2}$ o. 0 p. $\frac{1}{ab^2c^3}$

4. a. a^4 b. $\frac{1}{x^{15}}$ c. 1 d. $\frac{1}{m^5w^3}$ e. $\frac{1}{x^{10}}$
f. $\frac{1}{a^{15}}$ g. c^{12} h. $\frac{1}{b^2}$ i. $\frac{1}{a^5x^5}$ j. $\frac{1}{x^3y^3z^3}$
k. $\frac{1}{x+y}$ l. $\frac{1}{a^2-2ab+b^2}$ m. $\frac{1}{x^8}$ n. z^9
o. a^3 p. $\frac{1}{y^5}$ q. $\frac{b^4}{a^4}$ r. $\frac{x}{y}$ s. $\frac{b^2+2bc+c^2}{a^2}$
t. 1

5. a. $6x^3$ b. $-\frac{6}{y^8}$ c. $-\frac{4}{a^4}$ d. u^4 e. $\frac{1}{w^{42}}$ f. $a^{10}b^{15}$
g. $\frac{t^9}{u^{12}}$ h. x^6 i. y^5 j. a^6b^{15} k. $\frac{1}{c^{15}d^{20}}$ l. 1

6. a. 23,000,000 b. 0.0000711 c. 0.0000000000509

7. a. $-\frac{1}{16}$ b. $\frac{1}{16}$ **8**. a. $\frac{125}{64}$ b. 9

9. a. $\frac{b^3}{a^2}$ b. $\frac{1}{x^{18}}$ **10**. a. $\frac{1}{a^5}$ b. $\frac{a-1}{a^3}$

11. a. $\frac{1}{x^{50}}$ b. z^{25} **12**. $\frac{1}{48x^3y^{18}}$

13. a. $\frac{1}{x^4}$ b. $a^{15}b^{20}$ **14**. a. $\frac{1}{u^2w^2}$ b. $\frac{1}{u^2+2uw+w^2}$

15. Because $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ which is Undefined.

16. 0.00000000000049

12

“A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing.”

– ***George Bernard Shaw***

CH XX – ADVANCED FACTORING

We can now factor lots of quadratic binomials (like $4x^2 - 9$) and trinomials (like $n^2 + 10n + 21$). Sorry to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor 4th degree expressions (an exponent of 4), and expressions containing GCFs you might never have seen before.

☐ *FACTORING QUARTICS*

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

$= (c^2 + 16)(c^2 - 16)$ (difference of squares)

$= \boxed{(c^2 + 16)(c + 4)(c - 4)}$ (difference of squares again)

Note: $c^2 + 16$ can<u>not</u> be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

$= (9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer consisting of four factors:

$$\boxed{(3a + 1)(3a - 1)(a + 2)(a - 2)}$$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$	b. $x^4 - x^2 - 6$
c. $n^4 - 10n^2 + 9$	d. $a^4 - 81$
e. $36w^4 - 25w^2 + 4$	f. $9x^4 - 34x^2 + 25$
g. $c^4 - 16$	h. $x^4 - 8x^2 - 9$
i. $x^4 - 3x^2 - 10$	j. $g^4 - 256$
k. $36u^4 - 85u^2 + 9$	l. $y^4 + 81$

☐ *THE GCF REVISITED*

<u>EXAMPLE 2:</u> **Factor: $(a + b)^2 + 4(a + b)$**

<u>Solution:</u> This problem consists of two terms: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a <u>single term</u> consisting of two factors:

$$\boxed{(a + b)(a + b + 4)}$$

It's a single term because the final operation is mulitplication.

NOTE: The thing not to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$$(a+b)^2 + 4(a+b) = a^2 + 2ab + b^2 + 4a + 4b$$

Do you really want to try to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, that occurs multiple times in an expression, it's usually best to leave it intact. Also, notice that we have converted a 2-termed expression into 1 term — we have factored.

Alternate Method: Let's try a substitution method. We might be able to better see the essence of the problem if we replace $a + b$ with a simpler symbol — for example, x can represent $a + b$. Then the original expression

$$(a+b)^2 + 4(a+b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x+4)$$

Now substitute in the reverse direction to get $a + b$ back in the problem:

$$\boldsymbol{(a+b)(a+b+4)} \qquad \text{(the same answer as before)}$$

EXAMPLE 3: **Factor: $x^2(u-w) - 100(u-w)$**

Solution: The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u-w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

EXAMPLE 4: **Factor: $w^2(x + z) - 4w(x + z) + 3(x + z)$**

Solution: Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w - 3)(w - 1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x + z)(w - 3)(w - 1)}$$

Homework

2. Factor each expression:

a. $(x + y)^2 + 7(x + y)$

b. $(a - b)^2 - c(a - b)$

c. $x^2(c + d) + 5(c + d)$

d. $n^2(a - b) - 9(a - b)$

e. $x^2(a + 4) + 5x(a + 4) + 6(a + 4)$

f. $y^2(m + n) + 7y(m + n)$

g. $2x^2(a + b) + 3x(a + b) - 5(a + b)$

h. $4x^2(w + z) - 9(w + z)$

i. $(u - w)^2 - 9(u - w)$

j. $n^2(a + b) - 9n(a + b)$

k. $(t + r)y^2 - 100(t + r)$

l. $3ax^2 - 20ax - 7a$

☐ *GROUPING WITH FOUR TERMS*

EXAMPLE 5: **Factor:** $a^2 + ac + ab + bc$

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF in each pair) :

$$a(a + c) + b(a + c) \qquad \text{(notice that the GCF is } a + c\text{)}$$

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue — using our knowledge of the previous section — and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication ($xy = yx$), the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and you should get the original expression.

EXAMPLE 6: **Factor:** $x^3 - 7x^2 - 9x + 63$

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier — should we choose 9 or –9? Ultimately, it's a trial-and-error process. Watch what happens if we choose –9 for the GCF:

$$x^2(x - 7) - 9(x - 7) \qquad \text{(check the signs carefully)}$$

We now see two terms whose GCF is $x - 7$. Pull it to the front:

$$(x - 7)(x^2 - 9)$$

All this, and we're still not done. The second factor is the difference of two squares — now we're done:

$$\boxed{(x - 7)(x + 3)(x - 3)}$$

EXAMPLE 7: **Factor:** $ab + cd + ad + bc$

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$$(ab + cd) + (ad + bc)$$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$$(ab + ad) + (cd + bc)$$

Pull out the GCF from each set of parentheses:

$$a(b + d) + c(d + b)$$

Do we have a common factor in these two terms? Well, does $b + d = d + b$? Since addition is commutative, they are equal. So the GCF is $b + d$, and when we pull it out in front, we're done:

$$\boxed{(b + d)(a + c)}$$

EXAMPLE 8: **Factor: $2ax - bx - 2ay + by$**

Solution: Group in pairs, as usual:

$$(2ax - bx) + (-2ay + by)$$

Pull out the GCF in each grouping:

$$x(2a - b) + y(-2a + b)$$

Problem: There's no common factor; however, the factors $2a - b$ and $-2a + b$ are opposites of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$$x(2a - b) - y(2a - b) \qquad \text{(distribute to make sure we're right)}$$

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)} \qquad \text{[Check by multiplying out]}$$

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$
b. $a^2 + ac + ab + bc$
c. $x^3 - 4x^2 + 3x - 12$
d. $n^3 - n^2 - 5n + 5$
e. $x^3 + x^2 - 9x - 9$
f. $ac - bd + bc - ad$

g. $xw + yz - xz - wy$
h. $2ac - 2ad + bc - bd$
i. $6xw - yz + 3xz - 2wy$
j. $hj - j^2 - hk + jk$
k. $ax + ay - bx - by$
l. $x^3 - 2x^2 - 25x + 50$
m. $xw + 2wy - xz - 2yz$
n. $a^3 - a^2 - 5a + 5$
o. $4tw - 2tx + 2w^2 - wx$
p. $6x^3 + 2x^2 - 9x - 3$
q. Not factorable
r. $6a^3 - 15a^2 + 10a - 25$

☐ MORE GROUPING AND SUBSTITUTION PROBLEMS

EXAMPLE 9: **Factor: $(w + z)^2 - a^2$**

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor: $x^2 + 6x + 9 - y^2$**

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$$x(x + 6) + (3 + y)(3 - y)$$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor -- a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$$(x^2 + 6x + 9) - y^2$$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$$(x + 3)^2 - y^2$$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x + 3 + y)(x + 3 - y)}$$

Homework

4. Factor each expression:

a. $(x + y)^2 - z^2$

b. $(a - b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u + w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

☐ *FACTORING CUBICS USING THE GCF*

EXAMPLE 11: **Factor each cubic (3rd degree) polynomial:**

A. $\mathbf{5q^3 + 10q^2 + 5q}$

This is not as bad as it looks, if we remember to start with the GCF:

$5q^3 + 10q^2 + 5q$ (the polynomial to factor)

$= 5q(q^2 + 2q + 1)$ (factor out $5q$, the GCF)

$= 5q(q + 1)(q + 1)$ (factor the trinomial)

$= \boxed{5q(q + 1)^2}$ (write it more simply)

B. $\mathbf{4x^3 - x}$

$= x(4x^2 - 1)$ (factor out x, the GCF)

$= \boxed{x(2x + 1)(2x - 1)}$ (difference of squares)

Homework

5. Factor each cubic polynomial:

a. $x^3 - x$

b. $2n^3 + 6n^2 + 4n$

c. $10a^3 - 5a^2 - 5a$

d. $7y^3 + 70y^2 + 175y$

e. $36w^3 - 9w$

f. $24z^3 - 20z^2 - 24z$

Review Problems

6. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$

j. $n^3 + 3n^2 - 16n - 48$

k. $(a + b)^2 - 5(a + b) + 6$

l. $(x - y)^2 + 7(x - y) + 6$

m. $(a - b)^2 + 6(a - b) - 16$

n. $hm - hn + km - kn$

Solutions

1. a. $(x^2 + 1)(x + 1)(x - 1)$

b. $(x^2 + 2)(x^2 - 3)$

c. $(n + 1)(n - 1)(n + 3)(n - 3)$

d. $(a^2 + 9)(a + 3)(a - 3)$

e. $(2w + 1)(2w - 1)(3w + 2)(3w - 2)$

f. $(x + 1)(x - 1)(3x + 5)(3x - 5)$

g. $(c^2 + 4)(c + 2)(c - 2)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 2)(x^2 - 5)$

j. $(g^2 + 16)(g +4)(g - 4)$

k. $(2u + 3)(2u - 3)(3u + 1)(3u - 1)$

l. Not factorable

2. a. $(x + y)(x + y + 7)$

b. $(a - b)(a - b - c)$

c. $(c + d)(x^2 + 5)$

d. $(a - b)(n + 3)(n - 3)$

e. $(a + 4)(x + 3)(x + 2)$

f. $y(m + n)(y + 7)$

g. $(a + b)(2x + 5)(x - 1)$

h. $(w + z)(2x + 3)(2x - 3)$

i. $(u - w)(u - w - 9)$

j. $n(a + b)(n - 9)$

k. $(t + r)(y + 10)(y - 10)$

l. $a(3x + 1)(x - 7)$

3. a. $(x + y)(w + z)$ b. $(a + b)(a + c)$ c. $(x^2 + 3)(x - 4)$
d. $(n^2 - 5)(n - 1)$ e. $(x + 1)(x + 3)(x - 3)$ f. $(a + b)(c - d)$
g. $(x - y)(w - z)$ h. $(2a + b)(c - d)$ i. $(3x - y)(2w + z)$
j. $(h - j)(j - k)$ k. $(a - b)(x + y)$ l. $(x - 2)(x + 5)(x - 5)$
m. $(x + 2y)(w - z)$ n. $(a^2 - 5)(a - 1)$ o. $(2t + w)(2w - x)$
p. $(2x^2 - 3)(3x + 1)$ q. Not factorable r. $(3a^2 + 5)(2a - 5)$

4. a. $(x + y + z)(x + y - z)$ b. $(a - b + c)(a - b - c)$
c. $(x + 2 + y)(x + 2 - y)$ d. $(n - 3 + Q)(n - 3 - Q)$
e. $(u + w + T)(u + w - T)$ f. $(y + 5 + x)(y + 5 - x)$
g. $(a + b + c)(a + b - c)$ h. $(w - y + 7)(w - y - 7)$
i. $(2x + 1 + t)\ (2x + 1 - t)$ j. $(3x - 2 + y)(3x - 2 - y)$

5. a. $x(x + 1)(x - 1)$ b. $2n(n + 1)(n + 2)$
c. $5a(2a + 1)(a - 1)$ d. $7y(y + 5)^2$
e. $9w(2w + 1)(2w - 1)$ f. $4z(3z + 2)(2z - 3)$

6. a. $10a(x^2 + 4)(x + 2)(x - 2)$ b. $(P - Q)(Z + 12)(Z - 12)$
c. $(2x - 3)(5x + 1)(5x - 1)$ d. $(3a + 2b)(4c - 5d)$
e. $(a - b + c)(a - b - c)$ f. $(x + y + 12)(x + y - 12)$
g. $(x + 5)(x - 5)(x + 3)(x - 3)$ h. $(x^2 + 1)(x + 3)(x - 3)$
i. $(x^2 + 9)(x - 7)$ j. $(n + 4)(n - 4)(n + 3)$
k. $(a + b - 3)(a + b - 2)$ l. $(x - y + 6)(x - y + 1)$
m. $(a - b + 8)(a - b - 2)$ n. $(m - n)(h + k)$

> *"A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."*
>
> Reverend Edward A. Malloy, *Monk's Reflections*

CH NN – COMPLETE FACTORING

Consider the problem of factoring **$10x^2 + 50x + 60$.** Look at the 10. Its factor pairs are 1 and 10, or 2 and 5. Now take a look at the 60. It's downright scary to consider all the pairs of factors of that number. But watch what happens in our first example if we deal with the GCF first, and then worry about the rest later.

☐ *COMPLETE FACTORING*

EXAMPLE 1: **Factor completely: $10x^2 + 50x + 60$**

Solution: As mentioned above, the secret to factoring this quadratic expression is to utilize the **GCF**. The variable x is not common to all three terms, so we'll ignore it. However, each of the three terms does contain a factor of 10 (since 10 divides evenly into each of the coefficients 10, 50, and 60). Thus,

$$
\begin{aligned}
& 10x^2 + 50x + 60 && \text{(the given expression)} \\
= \; & 10(x^2 + 5x + 6) && \text{(pull out the GCF of 10)} \\
= \; & 10(x + 3)(x + 2) && \text{(factor the quadratic)}
\end{aligned}
$$

Not so difficult, after all. Therefore, the complete factorization of $10x^2 + 50x + 60$ is

$$\boxed{10(x + 3)(x + 2)}$$

The key to <u>complete</u> factoring

is to FIRST pull out the GCF!

Homework

1. Factor each expression <u>completely</u>:

a. $7x^2 - 35x + 42$

b. $10n^2 - 10$

c. $5a^2 - 30a + 45$

d. $50u^2 - 25u - 25$

e. $7w^2 - 700$

f. $9n^2 + 9$

g. $5y^2 - 125$

h. $3x^2 + 15x + 12$

i. $14x^2 - 7x - 7$

j. $13t^2 + 117$

k. $48z^2 - 28z + 4$

l. $24a^2 - 120a + 150$

☐ *REDUCING FRACTIONS*

<u>EXAMPLE 2:</u> **Reduce to lowest terms:** $\dfrac{5a^2 - 45}{a^2 + 6a + 9}$

<u>Solution:</u> We need to factor the numerator and the denominator. If we then see any common factors, we can divide them out.

$$\frac{5a^2 - 45}{a^2 + 6a + 9} = \frac{5(a^2 - 9)}{a^2 + 6a + 9} = \frac{5(a+3)(a-3)}{(a+3)(a+3)} = \frac{5\cancel{(a+3)}(a-3)}{\cancel{(a+3)}(a+3)}$$

Notice that factoring the numerator required two steps: pulling out the GCF of 5, followed by factoring the $a^2 - 9$. If we hadn't factored out the 5, we would never have been able to divide out anything, and we would have reached the false conclusion that the fraction is not reducible. So we can write the reduced fraction as $\frac{5(a-3)}{a+3}$. But, assuming it's not too much work, it's customary to remove parentheses. Thus, the final answer (after distributing the 5 to the $a - 3$) is

$$\boxed{\frac{5a-15}{a+3}}$$

Homework

2. Reduce each fraction to lowest terms:

a. $\frac{n^2-4}{n^2-4n+4}$

b. $\frac{2x^2+8x+6}{6x^2+18x+12}$

c. $\frac{x^2-4x+1}{x^2-4x+1}$

d. $\frac{10y^2-30y+20}{5y^2-15y+10}$

e. $\frac{2x^2-2}{4x-4}$

f. $\frac{3n^2-3n-90}{3n^2+30n+75}$

g. $\frac{14x+98}{21x^2-63x-1470}$

h. $\frac{5a^2-30a-135}{10a^2-60a-270}$

☐ ADDITIONAL QUADRATIC EQUATIONS

Now we'll use the GCF factoring method to solve more quadratic equations. The following example should convince you that factoring out a simple number first makes the rest of the factoring — and thus solving the equation — vastly easier.

EXAMPLE 3: **Solve for k: $16k^2 = 40k + 24$**

Solution: Solving any quadratic equation by factoring requires making one side of the equation zero. To this end, we will first bring the 40k and the 24 to the left side, factor in two steps, divide each side by the greatest common factor, set each factor to 0, and then solve each resulting equation. (That's a lot of steps!)

$$
\begin{array}{lll}
 & 16k^2 = 40k + 24 & \text{(the original equation)} \\
\Rightarrow & 16k^2 - 40k - 24 = 0 & \text{(subtract } 40k \text{ and 24)} \\
\Rightarrow & 8(2k^2 - 5k - 3) = 0 & \text{(factor out 8, the GCF)} \\
\Rightarrow & \dfrac{\cancel{8}(2k^2 - 5k - 3)}{\cancel{8}} = \dfrac{0}{8} & \text{(divide each side by 8)} \\
\Rightarrow & 2k^2 - 5k - 3 = 0 & \text{(simplify)} \\
\Rightarrow & (2k + 1)(k - 3) = 0 & \text{(factor)} \\
\Rightarrow & 2k + 1 = 0 \quad \text{or} \quad k - 3 = 0 & \text{(set each factor to 0)} \\
\Rightarrow & \boxed{k = -\frac{1}{2} \quad \text{or} \quad k = 3} & \text{(solve each linear equation)}
\end{array}
$$

This was a lot of work — you might be thinking that the Quadratic Formula (assuming you've studied it) would have been easier to use than factoring, and you're probably right! Generally, you can use whichever method you prefer (assuming it works), but pay attention to the method your instructor may require on an exam.

❑ CAREFUL ! ! !

Do you see the step in the preceding example where we divided both sides of the equation by 8? This was legal because we did the same thing to both sides of the equation, and we did NOT divide by zero. Do not ever fall into the trap of dividing each side of an equation by something with the variable in it — you may unknowingly be dividing by zero, resulting in a loss of a solution to the equation.

To illustrate this warning , the correct way to solve the quadratic equation $x^2 + x = 0$ is the following:

$$x^2 + x = 0$$

$$\Rightarrow \quad x(x + 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow \quad \underline{x = 0} \text{ or } \underline{x = -1}$$

We have two solutions: 0 and −1, no doubt about it.

Check: $x = 0: \quad \mathbf{0}^2 + \mathbf{0} = 0 + 0 = 0$ ✓

$x = -1: \quad (\mathbf{-1})^2 + (\mathbf{-1}) = 1 + (-1) = 1 - 1 = 0$ ✓

Now let's do it the wrong way, by dividing by the variable:

$$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow \frac{\cancel{x}(x+1)}{\cancel{x}} = \frac{0}{x} \Rightarrow x+1 = 0$$

$\Rightarrow$ $x = -1$, which is merely one of the two solutions. That is, *we lost a solution when we divided by the variable.* Since the purpose of algebra is to obtain solutions — not throw them away — we see that dividing by the variable was a really bad idea.

Homework

3. Solve each quadratic equation:

a. $7x^2 - 35x + 42 = 0$

b. $10n^2 = 10$

c. $5a^2 + 45 = 30a$

d. $50u^2 = 25u + 25$

e. $7w^2 - 700 = 0$

f. $180z^2 - 30z - 60 = 0$

g. $4x^2 + 4x - 24 = 0$

h. $16x^2 = 6 - 4x$

i. $10x^2 - 490 = 0$

j. $75w^2 + 48 = 120w$

Review Problems

4. Factor completely: $30q^2 + 68q + 30$

5. Reduce: $\dfrac{18x+18}{14x^2+42x+28}$

6. Solve for x by factoring: $30x^2 = 190x + 140$

☐ *To ∞ and Beyond*

A. Factor: $x^2 - 363x + 21{,}600$

Hint: Use the Quadratic Formula (and your calculator) to solve the equation $x^2 - 363x + 21{,}600 = 0$, and then use the solutions to write the factorization.

B. Solve for x: $x^4 - 13x^2 + 36 = 0$

Hint: Factoring will take two steps, producing four solutions.

Solutions

1. a. $7(x-3)(x-2)$ b. $10(n+1)(n-1)$ c. $5(a-3)^2$
d. $25(2u+1)(u-1)$ e. $7(w+10)(w-10)$ f. $9(n^2+1)$
g. $5(y+5)(y-5)$ h. $3(x+1)(x+4)$ i. $7(2x+1)(x-1)$
j. $13(t^2+9)$ k. $4(4z-1)(3z-1)$ l. $6(2a-5)^2$

2. a. $\frac{n+2}{n-2}$ b. $\frac{x+3}{3x+6}$ c. 1 d. 2
e. $\frac{x+1}{2}$ f. $\frac{n-6}{n+5}$ g. $\frac{2}{3x-30}$ h. $\frac{1}{2}$

3. a. 2, 3 b. ±1 c. 3 d. $1, -\frac{1}{2}$ e. ±10
f. $\frac{2}{3}, -\frac{1}{2}$ g. 2, –3 h. $\frac{1}{2}, -\frac{3}{4}$ i. ±7 j. $\frac{4}{5}$

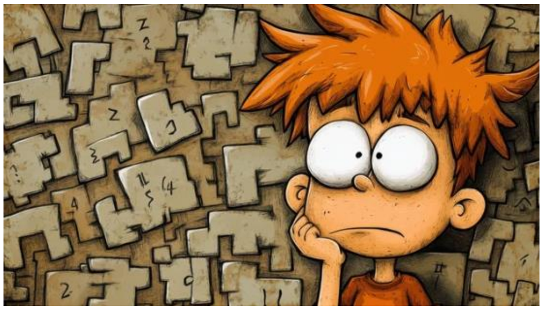
4. $2(5x+3)(3x+5)$ **5.** $\frac{9}{7x+14}$ **6.** $7, -\frac{2}{3}$

– Definition of **_happiness_** by John F. Kennedy (1917-1963)

“The full use of your powers along lines of excellence.”

Ch nn – Complete Factoring

CH NN – FACTORING THE SUM AND DIFFERENCE OF CUBES



We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored.

Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored — and perhaps surprisingly — even the ***sum of cubes*** $x^3 + y^3$ can be factored.

☐ *DIVISION, REMAINDERS, AND FACTORS*

Is 6 a **factor** of 161? **No** — divide 161 by 6 and you'll get 26 remainder **5**. Since the remainder is not 0, 6 is not a factor of 161. In other words, 6 does not go into 161 "evenly."

Is 7 a **factor** of 161? **Yes** — divide 161 by 7 and you'll get 23, remainder **0**. Thus, 7 divides into 161 exactly 23 times. And therefore, 161 = 7 × 23. We have factored 161 into 7 × 23 by showing that the factor 7 divides into 161 without remainder.

These observations are the key to factoring the sum and difference of cubes.

☐ *PERFECT CUBES*

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

125 is a *perfect cube* because it's the cube of 5:

$$125 = 5^3$$

1 is a perfect cube because it's the cube of 1:

$$1 = 1^3$$

–27 is a perfect cube because it's the cube of –3:

$$27 = (-3)^3$$

0 is a perfect cube because it's the cube of 0:

$$0 = 0^3$$

$\boldsymbol{x^3}$ is a perfect cube because it's the cube of x:

$$x^3 = x^3$$

$\boldsymbol{27y^3}$ is a perfect cube because it's the cube of $3y$:

$$27y^3 = (3y)^3$$

$\boldsymbol{8n^6}$ is a perfect cube because it's the cube of $2n^2$:

$$8n^6 = (2n^2)^3$$

$\boldsymbol{64z^{12}}$ is a perfect cube because it's the cube of $4z^4$:

$$64z^{12} = (4z^4)^3$$

$\boldsymbol{(a - b)^3}$ is a perfect cube because it's the cube of $a - b$.

Homework

1. a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

☐ *FACTORING A SUM OF CUBES – LONG DIVISION*

We're now ready to try to factor a sum of cubes; for example, what is the factorization of

$x^3 + 8$? [What should we multiply to get $x^3 + 8$?]

To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of 0. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of x^3 and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$
\begin{array}{r|l}
 & \qquad\qquad x^2 \;\; - 2x + 4 \\
\cline{2-2}
x+2 & x^3 + 0x^2 + 0x + 8 \\
 & \underline{x^3 \;\; + 2x^2} \\
 & \qquad - 2x^2 + 0x \\
 & \qquad \underline{- 2x^2 - 4x} \\
 & \qquad\qquad\quad 4x + 8 \\
 & \qquad\qquad\quad \underline{4x + 8} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that $x + 2$ is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division as a multiplication problem, giving us the factorization of $x^3 + 8$:

$$\boxed{x^3+8 \;=\; (x+2)(x^2-2x+4)}$$

Check by multiplying

❑ *FACTORING A DIFFERENCE OF CUBES – LONG DIVISION*

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of n^3 and -27, namely n and -3. This time it's your turn to carry out the long division. Here's what you should end up with:

$$\begin{array}{r|l} \multicolumn{1}{r}{} & n^2+3n+9 \\ \cline{2-2} n-3 & n^3+0n^2+0n-27 \end{array}$$

We now have our factorization:

$$\boxed{n^3-27 \;=\; (n-3)(n^2+3n+9)}$$

Again, check by multiplying

<u>EXAMPLE 1:</u>

A. Factor: $N^3 - 1$

Divide $N^3 - 1$ by $N - 1$ and you should get the factorization

$$\mathbf{N^3 - 1 \;=\; (N - 1)(N^2 + N + 1)}$$

B. Factor: $8p^3 + 27$

Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, giving

$$\mathbf{8p^3 + 27 \;=\; (2p + 3)(4p^2 - 6p + 9)}$$

C. Factor: $(a + b)^3 - 125$.

This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The appropriate quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any x's in it. So we need to substitute back the other way — when we convert each x back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\boldsymbol{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}$$

Homework

2. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

3. Factor each expression:

   a. $x^3 - 8$ b. $n^3 + 27$ c. $z^3 + 1$

   d. $8x^3 - 27$ e. $27y^3 + 125$ f. $64a^3 - 1$

4. Factor each expression:

   a. $(x + y)^3 + 8$ b. $(a - b)^3 - 27$ c. $(p + q)^3 + 1$

5. Factor $x^5 + 1$. Hint: Divide by $x + 1$.

6. Factor $A^5 - 32$. Hint: Divide by $A - 2$.

7. Factor each expression:

   a. $w^5 + 1$ b. $c^5 - 1$ c. $y^5 - 32$

   d. $z^5 + 32$ e. $n^5 + 243$ f. $m^5 - 243$

8. Factor each expression:

   a. $x^7 - 1$ b. $y^7 + 1$ c. $u^7 - 128$ d. $z^7 + 128$

☐ *CREATING FORMULS FOR FACTORING CUBES*

Factoring the sum (or difference) of cubes using long division is kind of tedious (and prone to errors). I suggest we do long division on a "generic" sum of cubes and see if we can come up with a formula that will allow us to factor the sum of cubes more efficiently.

Let's start by considering the factorization of $x^3 + y^3$. If you've understood this chapter so far, you'll understand what I'm about to do: Divide $x^3 + y^3$ by $x + y$, and confirm that the remainder comes out to be 0. Then we'll have our factorization.

$$x + y \,\overline{\big)\, x^3 + y^3}$$

We first have to pad up the dividend with some placeholders. What terms to include may not be obvious, but there's a nice pattern here: The powers on the x's go down while the powers on the y's go up.

$$
\begin{array}{rl}
 & x^2 - xy + y^2 \\
x+y & \overline{)\,x^3 + 0x^2y + 0xy^2 + y^3} \\
 & \underline{x^3 + x^2y} \\
 & \qquad -x^2y + 0xy^2 \\
 & \qquad \underline{-x^2y - xy^2} \\
 & \qquad\qquad\quad xy^2 + y^3 \\
 & \qquad\qquad\quad \underline{xy^2 + y^3} \\
 & \qquad\qquad\qquad\qquad \mathbf{0} \checkmark
\end{array}
$$

Since the remainder is 0, we have our factorization:

$$\boxed{x^3 + y^3 = (x+y)(x^2 - xy + y^2)}$$

EXAMPLE 2: **Factor: $a^3 + 8$**

$a^3 + 8$

$= \quad a^3 + 2^3$ [Here $x = a$ and $y = 2$]

$= \quad (a + 2)(a^2 - a \cdot 2 + 2^2)$ [From the formula for $x^3 + y^3$]

$= \quad \boxed{(a + 2)(a^2 - 2a + 4)}$ [Multiply out to confirm]

EXAMPLE 3: **Factor: $n^6 + 125w^3$**

$n^6 + 125w^3$

$= \quad (n^2)^3 + (5w)^3$ [Here $x = n^2$ and $y = 5w$]

$= \quad (n^2 + 5w)\left((n^2)^2 - n(5w) + (5w)^2\right)$

$= \quad \boxed{(n^2 + 5w)\left(n^4 - 5nw + 25w^2\right)}$

Homework

9. Find the factorization of $x^3 - y^3$ by dividing it by $x - y$.

10. Redo Problem #3 using our two new formulas for factoring the sum and difference of cubes.

☐ *To ∞ and Beyond*

A. Consider the expression $\boldsymbol{x^6 - y^6}$. Factor it in two ways:

1) Factor it as a difference of squares, followed by using the sum and difference of cubes.

2) Factor it as a difference of cubes, followed by a difference of squares.

Which method do you think is better, and why?

B. If I asked you for the *cube root* of 8, you'd probably say 2. And you're right, $2^3 = 8$. ✓ But if you have studied imaginary numbers (involving the number ***i***, defined by $\boldsymbol{i} = \sqrt{-1}$), then the fact is that 8 actually has three cube roots, the real root 2 we saw above, and two *non-real* roots (roots containing the imaginary number ***i***).

Here's how we can find them. Let x represent a cube root of 8. This implies that $x^3 = 8$, which implies that $x^3 - 8 = 0$. We now have a cubic equation that just might produce three solutions (just like a quadratic equation (degree = 2) might yield two solutions). These three potential solutions will all be cube roots of 8. Here's where the factoring in the chapter comes in:

$$x^3 - 8 = 0$$

$$\Rightarrow \quad (x-2)(x^2+2x+4) = 0 \qquad \text{(difference of cubes)}$$

$$\Rightarrow \quad x-2=0 \text{ OR } x^2+2x+4 = 0 \qquad (AB=0 \Rightarrow A=0 \text{ OR } B=0)$$

The first equation (linear) is easy to solve: **x = 2**, which we already knew was a cube root of 8.

To solve the second equation (quadratic) we will apply the Quadratic Formula (or we could complete the square). Since $a = 1$, $b = 2$, and $c = 4$, we can solve for x:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}\,i}{2} \qquad \text{(the } i \text{ is \underline{not} in the radical sign)}$$

$$= \frac{2\left(-1 \pm \sqrt{3}\,i\right)}{2} \qquad \text{(factor the numerator)}$$

$$= -1 \pm \sqrt{3}\,i \qquad \text{(divide out the 2's, and notice that we have two solutions)}$$

And we now have our three solutions to the equation $x^3 - 8 = 0$, which is equivalent to having our three cube roots of 8, as promised:

$$\boxed{2,\ -1+\sqrt{3}\,i,\ \text{and}\ -1-\sqrt{3}\,i}$$

If we allow ourselves to venture beyond the real numbers, we can find <u>three</u> cube roots of 8.

If you're up to it, verify the two complex cube roots of 8 by cubing each one and verifying that the final result will be 8 in both cases.

Solutions

1. a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

2. a. $(x + 2)(x^2 - 2x + 4) = x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 = x^3 + 8$ ✓

b. You try it.

3. a. $(x - 2)(x^2 + 2x + 4)$ b. $(n + 3)(n^2 - 3n + 9)$

c. $(z + 1)(z^2 - z + 1)$ d. $(2x - 3)(4x^2 + 6x + 9)$

e. $(3y + 5)(9y^2 - 15y + 25)$ f. $(4a - 1)(16a^2 + 4a + 1)$

4. a. $(x + y + 2)(x^2 + 2xy + y^2 - 2x - 2y + 4)$

b. $(a - b - 3)(a^2 - 2ab + b^2 + 3a - 3b + 9)$

c. $(p + q + 1)(p^2 + 2pq + q^2 - p - q + 1)$

5. $(x + 1)(x^4 - x^3 + x^2 - x + 1)$

6. $(A - 2)(A^4 + 2A^3 + 4A^2 + 8A + 16)$

7. a. $(w + 1)(w^4 - w^3 + w^2 - w + 1)$

b. $(c - 1)(c^4 + c^3 + c^2 + c + 1)$

c. $(y - 2)(y^4 + 2y^3 + 4y^2 + 8y + 16)$

d. $(z + 2)(z^4 - 2z^3 + 4z^2 - 8z + 16)$

e. $(n + 3)(n^4 - 3n^3 + 9n^2 - 27n + 81)$

f. $(m - 3)(m^4 + 3m^3 + 9m^2 + 27m + 81)$

8. a. $(x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

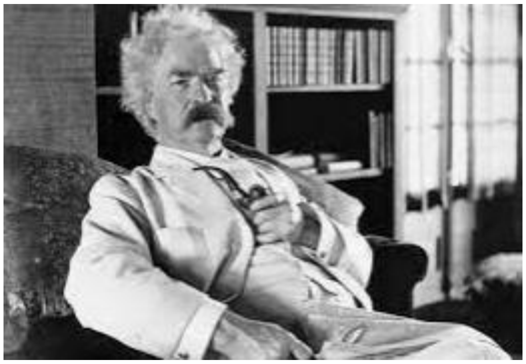
b. $(y + 1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1)$

c. $(u - 2)(u^6 + 2u^5 + 4u^4 + 8u^3 + 16u^2 + 32u + 64)$

d. $(z + 2)(z^6 - 2z^5 + 4z^4 - 8z^3 + 16z^2 - 32z + 64)$

9. $\boxed{x^3 - y^3 = (x - y)(x^2 + xy + y^2)}$

"Keep away from people who try to belittle your ambitions. Small people always do that, but the really great make you feel that you, too, can become great."



Mark Twain

CH XX – FRACTIONAL EQUATIONS

A ***fractional equation*** is an equation with one or more fractions in it. Since any fraction is really a division problem, it seems reasonable that we could solve such an equation by multiplying each side of the equation by something. What should that something be? The ***LCD*** (Least Common Denominator) of all the denominators in the equation works perfectly.

☐ SOLVING FRACTIONAL EQUATIONS

EXAMPLE 1: **Solve for z:** $\frac{z+6}{4} - \frac{z+4}{6} = \frac{2}{3}$

Solution:

First step: Determine the LCD; it's **12**. Reason: All three denominators, the 4, the 6, and the 3, divide into 12 — AND 12 is the smallest number that does this.

Second step: Multiply each side of the equation by the LCD. [If this is done correctly, all denominators will be eradicated.]

Third step: Distribute, cross-cancel, and solve the equation.

2

Multiply each side of the equation by 12, the LCD:

$$12\left(\frac{z+6}{4}-\frac{z+4}{6}\right) = 12\left(\frac{2}{3}\right)$$

Distribute (and write 12 as $\frac{12}{1}$ if you'd like):

$$\frac{12}{1}\left(\frac{z+6}{4}\right) - \frac{12}{1}\left(\frac{z+4}{6}\right) = \frac{12}{1}\left(\frac{2}{3}\right)$$

Cross-cancel, thus removing all fractions from the equation:

$${}^{3}\frac{\cancel{12}}{1}\left(\frac{z+6}{\cancel{4}_{1}}\right) - {}^{2}\frac{\cancel{12}}{1}\left(\frac{z+4}{\cancel{6}_{1}}\right) = {}^{4}\frac{\cancel{12}}{1}\left(\frac{2}{\cancel{3}_{1}}\right)$$

$$3(z+6) - 2(z+4) = 4(2)$$

Now solve for z in the usual way:

$$3z + 18 - 2z - 8 = 8$$

$$z + 10 = 8$$

$$\boxed{z = -2}$$

CHECK: Let $z = -2$ in the original equation. Do NOT do any algebra; just do simple arithmetic on each side:

$\frac{z+6}{4} - \frac{z+4}{6}$	$\frac{2}{3}$
$\frac{\mathbf{-2}+6}{4} - \frac{\mathbf{-2}+4}{6}$	
$\frac{4}{4} - \frac{2}{6}$	
$1 - \frac{1}{3}$	
$\frac{2}{3}$	✓

Homework

1. Solve and check each equation:

a. $\dfrac{m}{5}+\dfrac{m}{4} = \dfrac{6}{5}$

b. $\dfrac{b}{2}-\dfrac{b}{6} = \dfrac{1}{3}$

c. $\dfrac{n}{3}-\dfrac{n}{2} = 1$

d. $\dfrac{r}{2}-\dfrac{r}{3} = \dfrac{1}{3}$

2. Solve and check each equation:

a. $\dfrac{m-4}{6} + \dfrac{m+6}{5} = 2$

b. $\dfrac{b-4}{5} - \dfrac{b+6}{4} = 3$

c. $\dfrac{c+3}{3} + \dfrac{c+5}{2} = \dfrac{1}{3}$

d. $\dfrac{k+4}{2} - \dfrac{k-4}{3} = 1$

<u>EXAMPLE 2:</u> **Solve for x:** $\dfrac{2}{x}+\dfrac{y}{z} = w$

<u>Solution:</u> The LCD is xz (remember, the w can be written $\frac{w}{1}$). Therefore, we shall multiply each side of the equation by xz:

$$\frac{2}{x}+\frac{y}{z} = w \qquad \text{(the original equation)}$$

$$\Rightarrow \quad xz\left(\frac{2}{x}+\frac{y}{z}\right) = xz(w) \qquad \text{(multiply each side by } xz\text{, the LCD)}$$

$$\Rightarrow \quad \cancel{x}z\left(\frac{2}{\cancel{x}}\right)+x\cancel{z}\left(\frac{y}{\cancel{z}}\right) = xz(w) \qquad \text{(distribute and cross-cancel)}$$

$$\Rightarrow \quad 2z + xy = wxz \qquad \text{(simplify)}$$

$$\Rightarrow \quad xy - wxz = -2z \qquad \text{(}x\text{'s to the left; the rest to the right)}$$

$$\Rightarrow \quad x(y - wz) = -2z \qquad \text{(factor out the } x\text{, the GCF)}$$

$$\Rightarrow \quad \boxed{x = \frac{-2z}{y-wz}} \qquad \text{(divide each side by } y - wz\text{)}$$

Homework

3. Solve for x: $\frac{a}{b} - \frac{x}{a} = c$

4. Solve for y: $\frac{y}{t} + \frac{c}{d} = \frac{u}{w}$

5. Solve for n: $\frac{c}{h} + \frac{w}{n} = \frac{b}{x}$

6. Solve for z: $\frac{a}{z} - \frac{x}{n} = m$

7. Solve for c: $\frac{b}{m} + \frac{z}{c} = \frac{n}{w}$

8. Solve for n: $\frac{u}{n} + \frac{m}{t} = d$

EXAMPLE 3: **Solve for d:** $\mathbf{7 - \frac{6}{d-2} = \frac{1}{d-2}}$

Solution: The LCD is $d - 2$, so this is what we'll multiply each side of the equation by:

$$\left[\frac{d-2}{1}\right]\left(7 - \frac{6}{d-2}\right) = \left[\frac{d-2}{1}\right]\left(\frac{1}{d-2}\right)$$

$$\Rightarrow \quad (d-2)(7) - \cancel{(d-2)}\left(\frac{6}{\cancel{d-2}}\right) = \left(\frac{\cancel{d-2}}{1}\right)\left(\frac{1}{\cancel{d-2}}\right) \qquad \text{(distribute)}$$

$$\Rightarrow \quad 7(d-2) - 6 = 1 \qquad \text{(cross-cancel)}$$

$$\Rightarrow \quad 7d - 14 - 6 = 1 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 7d - 20 = 1 \qquad \text{(simplify)}$$

$$\Rightarrow \quad 7d = 21 \qquad \text{(add 20)}$$

$$\Rightarrow \quad \boxed{d = 3} \qquad \text{(divide by 7)}$$

CHECK: Put 3 in for d in the original equation:

$7-\frac{6}{d-2}$	$\frac{1}{d-2}$
$7-\frac{6}{\mathbf{3}-2}$	$\frac{1}{\mathbf{3}-2}$
$7-\frac{6}{1}$	$\frac{1}{1}$
$7-6$	1
1	✓

The sides balance, so $d = 3$ is the solution.

EXAMPLE 4: **Solve for u:** $\mathbf{1-\frac{16}{u}+\frac{28}{u^2} = 0}$

Solution: The LCD is u^2. Multiply each side of the equation by u^2:

$$u^2\left(1-\frac{16}{u}+\frac{28}{u^2}\right) = u^2(0)$$

$$\Rightarrow \quad u^2(1)-u^2\left(\frac{16}{u}\right)+u^2\left(\frac{28}{u^2}\right) = u^2(0) \qquad \text{(distribute)}$$

$$\Rightarrow \quad u^2-16u+28 = 0 \qquad \text{(cross-cancel)}$$

$$\Rightarrow \quad (u-14)(u-2) = 0 \qquad \text{(factor)}$$

$$\Rightarrow \quad u-14 = 0 \text{ or } u-2 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad \boxed{u = 14 \text{ or } u = 2}$$

You should check these two potential solutions to ensure that both work in the original equation.

EXAMPLE 5: **Solve for m:** $\dfrac{m}{m+9} + 8 = \dfrac{-9}{m+9}$

Solution: We begin as usual by multiplying each side of the equation by the LCD; in this problem, it's $m + 9$:

$$[m+9]\left(\frac{m}{m+9}+8\right) = [m+9]\left(\frac{-9}{m+9}\right)$$

$$\Rightarrow \quad [m+9]\frac{m}{m+9}+[m+9]8 = [m+9]\frac{-9}{m+9} \qquad \text{(distribute)}$$

$$\Rightarrow \quad \cancel{[m+9]}\frac{m}{\cancel{m+9}}+[m+9]8 = \cancel{[m+9]}\frac{-9}{\cancel{m+9}}$$

$$\Rightarrow \quad m + 8m + 72 = -9 \qquad \text{(cross-cancel)}$$

$$\Rightarrow \quad 9m + 72 = -9 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 9m = -81 \qquad \text{(subtract 72)}$$

$$\Rightarrow \quad \underline{m = -9} \qquad \text{(divide by 9)}$$

Now for the **check**; put –9 in for m in the original equation:

$$\begin{array}{c|c}
\dfrac{m}{m+9}+8 & \dfrac{-9}{m+9} \\[2ex]
\dfrac{\mathbf{-9}}{\mathbf{-9}+9}+8 & \dfrac{-9}{\mathbf{-9}+9} \\[2ex]
\dfrac{-9}{0}+8 & \dfrac{-9}{0}
\end{array}$$

Hold it right there!! We have zero on the bottom of both fractions. Continuing with this calculation is meaningless, since we have two undefined fractions. We've reached the end of our check. Does this mean we made a mistake in solving the equation? Absolutely not! We did everything according to the rules, so the equation must be inherently flawed. Thus, our only candidate for a solution, –9, has failed to be a solution. We have no choice but to admit that no number will make the original equation true. Our conclusion:

$\boxed{\text{No solution}}$ (Some teachers would write this as the null set: ∅)

EXAMPLE 6: **Solve for x:** $\dfrac{5}{x+1} - \dfrac{3}{x+3} = 4$

Solution: Multiply each side of the equation by the LCD: $(x + 1)(x + 3)$. Then distribute, cross-cancel, and solve the resulting quadratic equation by factoring.

Multiply each side of the equation by the LCD, $(x + 1)(x + 3)$:

$$(x+1)(x+3)\left(\frac{5}{x+1} - \frac{3}{x+3}\right) = (x+1)(x+3)(4)$$

Distribute the LCD, and then cross-cancel:

$$\cancel{(x+1)}(x+3)\left(\frac{5}{\cancel{x+1}}\right) - (x+1)\cancel{(x+3)}\left(\frac{3}{\cancel{x+3}}\right) = (x+1)(x+3)(4)$$

$$5(x + 3) - 3(x + 1) = 4(x + 1)(x + 3)$$

Distribute on the left and double distribute on the right:

$$5x + 15 - 3x - 3 = 4(x^2 + 4x + 3)$$

Combine like terms on the left, and distribute on the right:

$$2x + 12 = 4x^2 + 16x + 12$$

Since we have a quadratic equation, bring all terms to one side:

$$0 = 4x^2 + 14x$$

Turn the equation around and factor:

$$2x(2x + 7) = 0$$

Set each factor to 0:

$$2x = 0 \text{ or } 2x + 7 = 0$$

Solve each linear equation:

$$\boxed{x = 0 \text{ or } x = -\frac{7}{2}}$$

CHECK: I'm tired; you check both solutions.

EXAMPLE 7: **Solve for *y*:** $\dfrac{y-2}{y+1} = \dfrac{y+7}{y-8}$

Solution: Multiply each side of the equation by the LCD:

$$(y+1)(y-8)\left(\frac{y-2}{y+1}\right) = (y+1)(y-8)\left(\frac{y+7}{y-8}\right)$$

Cross-cancel the common factors on each side of the equation:

$$\cancel{(y+1)}(y-8)\left(\frac{y-2}{\cancel{y+1}}\right) = (y+1)\cancel{(y-8)}\left(\frac{y+7}{\cancel{y-8}}\right)$$

And simplify:

$$(y-8)(y-2) = (y+1)(y+7)$$

Notice that this equation can be found directly from the original equation by "cross-multiplying."

Double distribute on each side of the equation:

$$y^2 - 10y + 16 = y^2 + 8y + 7$$

Subtract y^2 from each side of the equation:

$$-10y + 16 = 8y + 7$$

Finally, solve for *y* in the standard way:

$$-18y + 16 = 7 \Rightarrow -18y = -9 \Rightarrow y = \frac{-9}{-18} = \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}}$$

CHECK:

$\frac{y-2}{y+1}$	$\frac{y+7}{y-8}$
$\frac{\frac{\mathbf{1}}{\mathbf{2}}-2}{\frac{\mathbf{1}}{\mathbf{2}}+1}$	$\frac{\frac{\mathbf{1}}{\mathbf{2}}+7}{\frac{\mathbf{1}}{\mathbf{2}}-8}$
$\frac{-\frac{3}{2}}{\frac{3}{2}}$	$\frac{\frac{15}{2}}{-\frac{15}{2}}$
-1	-1 ✓

Homework

9. Solve and check each equation:

a. $\frac{4x}{x-8}+1 = \frac{9}{x-8}$

b. $\frac{1}{2n}+\frac{2}{n} = \frac{5}{6}$

c. $1-\frac{12}{a}-\frac{64}{a^2} = 0$

d. $g+\frac{24}{g} = -10$

e. $\frac{4}{t+1}-\frac{6}{t+3} = 2$

f. $\frac{9}{b-1}-\frac{9}{b+3} = 3$

g. $\frac{x+2}{x+4} = \frac{x+5}{x}$

h. $\frac{q-5}{-2} = \frac{-6}{q-1}$

i. $\frac{9}{x}-\frac{70}{x^2} = -1$

j. $\frac{-8}{g+1} = \frac{8g}{g+1}-9$

k. $\frac{1}{4a}+\frac{2}{5a} = \frac{8}{5}$

l. $\frac{2}{m+8}+\frac{1}{m+6} = 1$

Review Problems

10. Solve for x: $\frac{x-2}{5} - \frac{2x-1}{6} = \frac{4}{15}$

11. Solve for x: $\frac{a}{y} - \frac{z}{x} = \frac{a}{b}$

12. Solve for a: $1 - \frac{25}{a} + \frac{150}{a^2} = 0$

13. Solve for x: $\frac{x}{x-3} + \frac{3}{2} = \frac{3}{x-3}$

14. Solve for x: $\frac{2}{x-3} + \frac{3}{x+1} = \frac{3}{2}$

Solutions

1. a. $m = \frac{8}{3}$ b. $b = 1$ c. $n = -6$ d. $r = 2$

2. a. $m = 4$ b. $b = -106$ c. $c = -\frac{19}{5}$ d. $k = -14$

3. $x = \frac{abc - a^2}{-b} = \frac{a^2 - abc}{b}$

4. $y = \frac{dtu - ctw}{dw}$

5. $n = \frac{-hxw}{cx - bh} = \frac{hxw}{bh - cx}$

6. $z = \frac{an}{mn + x}$

7. $c = \frac{-mwz}{bw - mn} = \frac{mwz}{mn - bw}$

8. $n = \frac{tu}{dt - m}$

9. a. $x = \frac{17}{5}$ b. $n = 3$ c. $a = 16, -4$

d. $g = -4, -6$ e. $t = 0, -5$ f. $b = 3, -5$

g. $x = -\frac{20}{7}$ h. $q = -1, 7$ i. $x = 5, -14$

j. No solution k. $a = \frac{13}{32}$ l. $m = -4, -7$

10. $x = -\frac{15}{4}$ **11**. $x = \frac{byz}{ab - ay}$

12. $a = 10,\ 15$ **13**. No solution **14**. $x = 5, \frac{1}{3}$

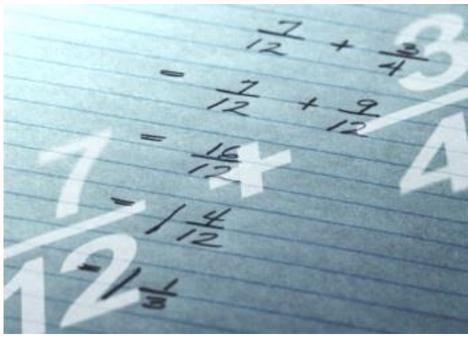
12

“We cannot hold a torch
to light another's path
without brightening our own.”

– Ben Sweetland

CH XX – ADDING AND SUBTRACTING FRACTIONS, PART I

A prior chapter, *Algebraic Fractions, an Introduction*, showed us that to add or subtract fractions, we must ensure that the denominators are the same. So, even though the fractions in this chapter are a little more challenging than we've seen before, the same rules and techniques apply.



❑ *MORE ADDING AND SUBTRACTING*

EXAMPLE 1: **Add:** $\boldsymbol{\frac{4u-21}{u+7} + \frac{u^2}{u+7}}$

Solution: We are adding fractions with the same denominator (u + 7), so all we have to do is add the numerators. The common denominator becomes the denominator of the sum:

$$\frac{4u-21+u^2}{u+7}$$

We've added the two fractions so that it's now a single fraction, but all fractional answers need to be reduced — we'll rearrange the terms in the numerator so it's easier to factor:

$$\frac{u^2+4u-21}{u+7}$$

Factor the numerator and divide out the common factor:

$$\frac{(u+7)(u-3)}{u+7} = \frac{{}^{1}\cancel{(u+7)}(u-3)}{\cancel{u+7}_{1}} = \boxed{u-3}$$

EXAMPLE 2: **Subtract:** $\displaystyle\frac{\mathbf{12x-10}}{\mathbf{x^2-2x-8}} - \frac{\mathbf{11x-12}}{\mathbf{x^2-2x-8}}$

Solution: Remember that both addition and subtraction of fractions require a *common denominator*. The fractions in this example already have the same denominator, so there's nothing to worry about here.

The tricky part is subtracting the numerators. We must subtract the second numerator (all of it!) from the first numerator — this is where parentheses come to the rescue:

$$\frac{12x-10}{x^2-2x-8} - \frac{11x-12}{x^2-2x-8}$$ (the original problem)

$$= \frac{(12x-10)-(11x-12)}{x^2-2x-8}$$ (one numerator minus all of the other)

$$= \frac{12x-10-11x+12}{x^2-2x-8}$$ (remove the parentheses, carefully)

$$= \frac{x+2}{x^2-2x-8}$$ (combine like terms)

$$= \frac{x+2}{(x+2)(x-4)}$$ (factor the denominator)

$$= \frac{\cancel{x+2}^{\,1}}{{}_{1}\cancel{(x+2)}(x-4)}$$ (divide out the common factor)

$$= \boxed{\frac{1}{x-4}}$$ (note that the numerator is 1)

EXAMPLE 3: **Subtract:** $\dfrac{10}{a^6} - \dfrac{7}{a^2}$

Solution: The denominators are different, so we can't subtract the fractions yet. To make the denominators the same, the second denominator (a^2) needs to be built up to match the first denominator (a^6). We ask, what should we multiply a^2 by to get a^6? Since a^2 is two factors of a, and a^6 is six factors of a, we would need four more factors of a to make them match. So we'll multiply the second fraction, top and bottom, by a^4.

$$\frac{10}{a^6} - \frac{7}{a^2}$$

$$= \frac{10}{a^6} - \frac{7 \times \boldsymbol{a^4}}{a^2 \times \boldsymbol{a^4}}$$

$$= \underbrace{\frac{10}{a^6} - \frac{7a^4}{a^6}}_{\text{same denominator}}$$

$$= \boxed{\frac{10 - 7a^4}{a^6}}$$

EXAMPLE 4: **Add:** $\dfrac{2}{x^2y^3} + \dfrac{7}{y^7z}$

Solution: This problem also requires a common denominator. To make the denominators the same, we notice that the first denominator needs four more factors of y (so that there will be seven of them) and a factor of z; the second denominator needs an x^2 factor in it. Here's how we do it:

$$\frac{2}{x^2y^3} + \frac{7}{y^7z} \qquad \text{(the original problem)}$$

$$= \frac{2 \bullet y^4 z}{x^2 y^3 \bullet y^4 z} + \frac{7 \bullet x^2}{y^7 z \bullet x^2} \qquad \text{(build to an LCD)}$$

$$= \underbrace{\frac{2y^4 z}{x^2 y^7 z} + \frac{7x^2}{x^2 y^7 z}}_{\text{same denominator}} \qquad \text{(simplify tops and bottoms)}$$

Now the denominators are the same, so we add the numerators and put that result over the common denominator:

$$\boxed{\frac{2y^4 z + 7x^2}{x^2 y^7 z}}$$

EXAMPLE 5: **Subtract:** $\frac{6k+4}{3} - \frac{2k+3}{2}$

Solution: The least common denominator (LCD) is 6, so we'll multiply the top and bottom of the first fraction by 2 and the top and bottom of the second fraction by 3:

$$\frac{6k+4}{3} - \frac{2k+3}{2} \qquad \text{(the original problem)}$$

$$= \frac{\mathbf{2}(6k+4)}{\mathbf{2}(3)} - \frac{\mathbf{3}(2k+3)}{\mathbf{3}(2)} \qquad \text{(create the LCD of 6)}$$

$$= \frac{12k+8}{6} - \frac{6k+9}{6} \qquad \text{(simplify tops and bottoms)}$$

Notice the Parentheses!

$$= \frac{(12k+8)-(6k+9)}{6} \qquad \text{(combine into a single fraction)}$$

$$= \frac{12k+8-6k-9}{6} \qquad \text{(distribute)}$$

$$= \boxed{\frac{6k-1}{6}} \qquad \text{(combine like terms)}$$

Homework

1. Add or subtract:

a. $\frac{d^2}{d-9} - \frac{d+72}{d-9}$

b. $\frac{3p+1}{p^2-6p-27} - \frac{2p-2}{p^2-6p-27}$

c. $\frac{n^2}{n-6} - \frac{7n-6}{n-6}$

d. $\frac{16x+63}{x+7} + \frac{x^2}{x+7}$

e. $\frac{-w-9}{w^2+8w-48} - \frac{-2w-5}{w^2+8w-48}$

f. $\frac{-6a-11}{a^2-10a-24} - \frac{-7a+1}{a^2-10a-24}$

2. Add or subtract:

a. $\frac{8}{5w^6} - \frac{3}{2w^4}$

b. $\frac{3}{b^6} + \frac{9}{5b^3}$

c. $\frac{2}{k^2} + \frac{7}{4k^5}$

d. $\frac{1}{9w^2} - \frac{4}{5w^4}$

e. $\frac{2}{5r^2} + \frac{5}{4r^4}$

f. $\frac{1}{z^3} - \frac{7}{6z^6}$

3. Add or subtract:

a. $\frac{2}{c^2m^2} + \frac{3}{m^2w^6}$

b. $\frac{4}{a^5c^5} - \frac{2}{c^6v^2}$

c. $\frac{5}{a^3b^2} + \frac{5}{b^5m^5}$

d. $\frac{2}{b^3h^2} - \frac{6}{h^3x^4}$

e. $\frac{5}{a^3c^2} + \frac{5}{c^4s^6}$

f. $\frac{6}{k^3t^4} - \frac{4}{t^5v^6}$

4. Add or subtract:

a. $\dfrac{2z-9}{8}+\dfrac{2z-9}{5}$ b. $\dfrac{8b+6}{6}-\dfrac{6b-8}{2}$

c. $\dfrac{-9m-9}{9}-\dfrac{-5m-9}{7}$ d. $\dfrac{4s-9}{3}+\dfrac{5s-3}{3}$

e. $\dfrac{-7u+9}{5}-\dfrac{4u-1}{6}$ f. $\dfrac{-2k-5}{4}+\dfrac{9k+1}{5}$

Solutions

1. a. $d+8$ b. $\dfrac{1}{p-9}$ c. $n-1$

d. $x+9$ e. $\dfrac{1}{w+12}$ f. $\dfrac{1}{a+2}$

2. a. $\dfrac{16-15w^2}{10w^6}$ b. $\dfrac{15+9b^3}{5b^6}$ c. $\dfrac{8k^3+7}{4k^5}$

d. $\dfrac{5w^2-36}{45w^4}$ e. $\dfrac{8r^2+25}{20r^4}$ f. $\dfrac{6z^3-7}{6z^6}$

3. a. $\dfrac{2w^6+3c^2}{c^2m^2w^6}$ b. $\dfrac{4cv^2-2a^5}{a^5c^6v^2}$ c. $\dfrac{5b^3m^5+5a^3}{a^3b^5m^5}$

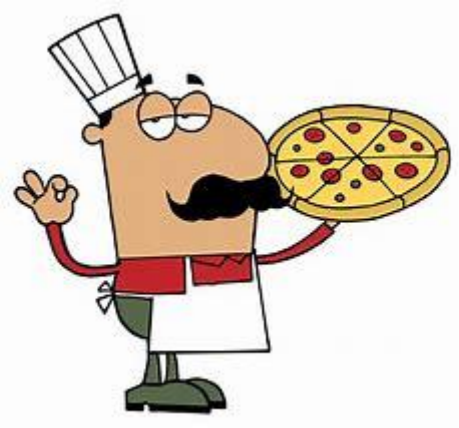
d. $\dfrac{2hx^4-6b^3}{b^3h^3x^4}$ e. $\dfrac{5c^2s^6+5a^3}{a^3c^4s^6}$ f. $\dfrac{6tv^6-4k^3}{k^3t^5v^6}$

4. a. $\dfrac{26z-117}{40}$ b. $\dfrac{-5b+15}{3}$ c. $\dfrac{-2m+2}{7}$

d. $3s-4$ e. $\dfrac{-62u+59}{30}$ f. $\dfrac{26k-21}{20}$

CH XX – ADDING AND SUBTRACTING FRACTIONS, PART II

❑ *INTRODUCTION*



We continue our analysis of algebraic fractions. Now the fractions will have more stuff in their numerators and denominators, but the basic rule still applies: To **add** or **subtract** fractions, we must have a common denominator.

❑ *MORE ADDING AND SUBTRACTING*

EXAMPLE 1: **Subtract:** $\boldsymbol{\frac{7}{x+3} - \frac{3}{x-2}}$

Solution: The two denominators have no factor in common. (Yes, they both have an x, but this is a term, not a factor.) The LCD is the product of the two denominators: $(x + 3)(x - 2)$. So we'll multiply the top and bottom of the first fraction by $(x - 2)$, and multiply the top and bottom of the second fraction by $(x + 3)$:

$$\frac{7}{x+3} - \frac{3}{x-2} \qquad \text{(the original problem)}$$

$$= \frac{7}{x+3}\underbrace{\left[\frac{x-2}{x-2}\right]}_{=1} - \frac{3}{x-2}\underbrace{\left[\frac{x+3}{x+3}\right]}_{=1} \qquad \text{(build up to the LCD)}$$

$$= \frac{7(x-2)}{(x+3)(x-2)} - \frac{3(x+3)}{(x-2)(x+3)} \qquad \text{(the bottoms are the same)}$$

Same denominator!

$$= \frac{7(x-2)-3(x+3)}{(x-2)(x+3)} \qquad \text{(combine the two fractions)}$$

$$= \frac{7x-14-3x-9}{x^2+x-6} \qquad \text{(distribute)}$$

$$= \boxed{\frac{4x-23}{x^2+x-6}} \qquad \text{(combine like terms)}$$

EXAMPLE 2: **Add:** $\boldsymbol{\dfrac{5}{m^2+5m+6} + \dfrac{3}{m+2}}$

Solution: This is a tricky one. To add these fractions, we have to multiply tops and bottoms by factors that will result in the LCD. But to know what factors to multiply by, we have to know what factors are already in the denominators. So, we factor the denominators:

$$\frac{5}{m^2+5m+6} + \frac{3}{m+2} = \frac{5}{(m+2)(m+3)} + \frac{3}{m+2}$$

Now we can easily see the factors in the denominators. To make the denominators the same, we need to multiply the top and bottom of the second fraction by $m + 3$. Then we can add the fractions:

$$\frac{5}{(m+2)(m+3)} + \frac{3}{m+2}\left[\frac{m+3}{m+3}\right] = \frac{5}{(m+2)(m+3)} + \frac{3(m+3)}{(m+2)(m+3)}$$

$$= \frac{5+3(m+3)}{(m+2)(m+3)} = \frac{5+3m+9}{m^2+5m+6} = \boxed{\frac{3m+14}{m^2+5m+6}}$$

EXAMPLE 3: **Subtract:** $\dfrac{8}{u^2+3u-4} - \dfrac{7-5u}{u^2-3u-28}$

Solution: First factor all the denominators:

$$\frac{8}{(u-1)(u+4)} - \frac{7-5u}{(u+4)(u-7)}$$

Now build up the fractions so that the LCD is created:

$$= \frac{8}{(u-1)(u+4)}\left[\frac{u-7}{u-7}\right] - \frac{7-5u}{(u+4)(u-7)}\left[\frac{u-1}{u-1}\right]$$

$$= \frac{8(u-7)}{(u-1)(u+4)(u-7)} - \frac{(7-5u)(u-1)}{(u-4)(u-7)(u-1)}$$

Notice that the denominators are the same.

$$= \frac{8(u-7)\ -\ (7-5u)(u-1)}{(u-1)(u+4)(u-7)}$$

(subtract the numerators)

Notice the parentheses.

$$= \frac{8u-56-(7u-7-5u^2+5u)}{(u-1)(u+4)(u-7)}$$

(distribute)

$$= \frac{8u-56-(-5u^2+12u-7)}{(u-1)(u+4)(u-7)}$$

(combine like terms)

$$= \frac{8u-56+5u^2-12u+7}{(u-1)(u+4)(u-7)}$$

(distribute the minus sign)

$$= \boxed{\frac{5u^2-4u-49}{(u-1)(u+4)(u-7)}}$$

(combine like terms)

You may wonder why we left the denominator in factored form. It's just a matter of opinion, but many teachers figure that the problem's hard enough without multiplying out those three binomials.

Homework

1. Perform the indicated operation:

a. $\frac{6}{d+1} + \frac{-4}{6d+5}$ b. $\frac{-6}{5h-2} - \frac{5}{4h-1}$

c. $\frac{-6}{q+1} + \frac{3}{5q+4}$ d. $\frac{3}{2t+5} - \frac{-6}{2t+1}$

2. Perform the indicated operation:

a. $\frac{3}{z-3} + \frac{1}{z^2+12z+32}$ b. $\frac{-3}{u^2+u-72} - \frac{3}{u+9}$

c. $\frac{-2}{w^2-19w+90} - \frac{2}{w-10}$ d. $\frac{4}{h+7} + \frac{-5}{h^2+11h+28}$

3. Perform the indicated operation:

a. $\frac{7}{b^2+5b-6} + \frac{3b-1}{b^2-4b-60}$ b. $\frac{-6}{a^2-16} - \frac{-5a+3}{a^2-5a+4}$

c. $\frac{-1}{s^2-s-42} - \frac{6s-7}{s^2-4s-21}$ d. $\frac{9}{v^2-9} + \frac{2v+1}{v^2+13v+30}$

Solutions

1. a. $\frac{32d+26}{(d+1)(6d+5)}$ or $\frac{32d+26}{6d^2+11d+5}$

b. $\frac{-49h+16}{(5h-2)(4h-1)}$

c. $\frac{-27q-21}{(q+1)(5q+4)}$

d. $\frac{18t+33}{(2t+5)(2t+1)}$

2. a. $\frac{3z^2+37z+93}{(z+4)(z+8)(z-3)}$

b. $\frac{-3u+21}{(u+9)(u-8)}$

c. $\frac{-2w+16}{(w-10)(w-9)}$

d. $\frac{4h+11}{(h+7)(h+4)}$

3. a. $\frac{3b^2+3b-69}{(b-1)(b+6)(b-10)}$

b. $\frac{5a^2+11a-6}{(a+4)(a-4)(a-1)}$

c. $\frac{-6s^2-30s+39}{(s+6)(s-7)(s+3)}$

d. $\frac{2v^2+4v+87}{(v-3)(v+3)(v+10)}$

"The illiterate of the 21st century will not be those who cannot read and write, but those who cannot learn, unlearn, and relearn."

– Alvin Toffler

MULTIPLYING AND DIVIDING FRACTIONS

EXAMPLE 1: $\frac{a^4k^5}{u^2b^2} \cdot \frac{a^3k^4}{u^2b}$

$= \frac{a^4k^5a^3k^4}{u^2b^2u^2b}$ (multiply tops and multiply bottoms)

$= \frac{a^4a^3k^5k^4}{u^2u^2b^2b}$ (rearrange the factors, top and bottom)

$= \boxed{\frac{a^7k^9}{u^4b^3}}$ (simplify by adding exponents)

EXAMPLE 2: $\frac{k^4u^6}{ws^2} \cdot \frac{ws^4}{k^6u}$

$= \frac{k^4u^6ws^4}{ws^2k^6u}$ (multiply tops and multiply bottoms)

$= \boxed{\frac{s^2u^5}{k^2}}$ (reduce the fraction)

Alternate solution, with factoring and cross-canceling:

$$\frac{k^4u^6}{ws^2} \cdot \frac{ws^4}{k^6u}$$

$$= \frac{k^4u^5u}{ws^2} \cdot \frac{ws^2s^2}{k^4k^2u}$$

$$= \frac{\cancel{k^4}u^5\cancel{u}}{\cancel{w}\cancel{s^2}} \cdot \frac{\cancel{w}s^2\cancel{s^2}}{\cancel{k^4}k^2\cancel{u}}$$

$$= \boxed{\frac{u^5s^2}{k^2}}$$

EXAMPLE 3: $\boldsymbol{\frac{x^2y}{z^2} \div \frac{y^4}{x^6z^5}}$

$$= \frac{x^2y}{z^2} \cdot \frac{x^6z^5}{y^4}$$ (multiply by the reciprocal)

$$= \boxed{\frac{x^8z^3}{y^3}}$$ (cross-cancel z's and y's)

EXAMPLE 4: **Multiply:** $\boldsymbol{\frac{x^2-9}{x^2-6x+9} \cdot \frac{x+2}{x^2+5x+6}}$

Solution:

$$\frac{x^2-9}{x^2-6x+9} \cdot \frac{x+2}{x^2+5x+6}$$ (the given problem)

$$= \frac{(x+3)(x-3)}{(x-3)(x-3)} \cdot \frac{x+2}{(x+2)(x+3)}$$ (factor everything)

$$= \frac{\cancel{(x+3)}\cancel{(x-3)}}{(x-3)\cancel{(x-3)}} \cdot \frac{\cancel{x+2}}{\cancel{(x+2)}\cancel{(x+3)}}$$ (cancel)

$$= \boxed{\frac{1}{x-3}}$$ (x – 3 is on the bottom)

EXAMPLE 5: **Divide:** $\dfrac{x+10}{x^2+6x-7} \div \dfrac{x-12}{x^2-13x+12}$

Solution:

$$\frac{x+10}{x^2+6x-7} \div \frac{x-12}{x^2-13x+12} \qquad \text{(the given problem)}$$

$$= \frac{x+10}{x^2+6x-7} \times \frac{x^2-13x+12}{x-12} \qquad \text{(invert and multiply)}$$

$$= \frac{x+10}{(x+7)(x-1)} \times \frac{(x-1)(x-12)}{x-12} \qquad \text{(factor everything)}$$

$$= \frac{x+10}{(x+7)\cancel{(x-1)}} \times \frac{\cancel{(x-1)}\,\cancel{(x-12)}}{\cancel{x-12}} \qquad \text{(cancel)}$$

$$= \boxed{\frac{x+10}{x+7}}$$

Homework

1. Multiply or divide:

a. $\dfrac{x^2k}{t^4u^4} \div \dfrac{t^4u^5}{x^4k^3}$ b. $\dfrac{r^6a^6}{v^5u} \div \dfrac{r^4a^4}{vu^6}$ c. $\dfrac{w^2r^5}{k^6x} \div \dfrac{kx^6}{w^2r^5}$

d. $\dfrac{r^5z^5}{b^2n^4} \cdot \dfrac{b^6n}{r^6z^4}$ e. $\dfrac{u^4w}{v^3m^2} \div \dfrac{uw^6}{v^6m}$ f. $\dfrac{ku^2}{v^6w^3} \cdot \dfrac{v^6w^5}{k^4u^3}$

2. Multiply or divide:

a. $\dfrac{c-2}{c^2} \cdot \dfrac{c^2+4c}{c+4}$ b. $\dfrac{m}{m^2-2m-35} \cdot \dfrac{m^2-m-42}{m+6}$

c. $\dfrac{c-2}{c^2} \div \dfrac{c-3}{c^2-4c}$ d. $\dfrac{v}{v^2+4v-21} \div \dfrac{v+10}{v^2-2v-63}$

e. $\frac{q+6}{q^2+6q} \div \frac{q}{q^2+6q}$

f. $\frac{v-1}{v^2-v-6} \cdot \frac{v^2-2v-3}{v+1}$

3. Multiply or divide:

a. $\frac{m-6}{m^2-21m+108} \div \frac{m+5}{m^2-4m-45}$

b. $\frac{c+7}{c^2-18c+80} \cdot \frac{c^2-6c-40}{c+4}$

c. $\frac{z-1}{z^2-15z+44} \cdot \frac{z^2+2z-24}{z+6}$

d. $\frac{z+4}{z^2+8z+7} \div \frac{z-4}{z^2-3z-4}$

e. $\frac{m+8}{m^2+22m+120} \div \frac{m+3}{m^2+18m+80}$

f. $\frac{u-8}{u^2-6u-7} \div \frac{u-10}{u^2-9u-10}$

Solutions

1. a. $\frac{x^6k^4}{t^8u^9}$ b. $\frac{r^2a^2u^5}{v^4}$ c. $\frac{w^4r^{10}}{k^7x^7}$

d. $\frac{zb^4}{rn^3}$ e. $\frac{u^3v^3}{w^5m}$ f. $\frac{w^2}{k^3u}$

2. a. $\frac{c-2}{c}$ b. $\frac{m}{m+5}$ c. $\frac{c^2-6c+8}{c^2-3c}$

d. $\frac{v^2-9v}{v^2+7v-30}$ e. $\frac{q+6}{q}$ f. $\frac{v-1}{v+2}$

3. a. $\frac{m-6}{m-12}$ b. $\frac{c+7}{c-8}$ c. $\frac{z-1}{z-11}$

d. $\frac{z+4}{z+7}$ e. $\frac{m^2+16m+64}{m^2+15m+36}$ f. $\frac{u-8}{u-7}$

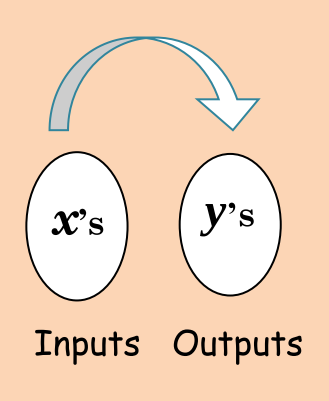
6

“Education is the power to think clearly, the power to act well in the world’s work, and the power to appreciate life.”

— *Brigham Young*

CH XX – FUNCTIONS: FORMULAS AND GRAPHS

An earlier chapter gave us the basic notion of a **function**: an entity where each input produces exactly one output. In this chapter we look at functions once again, this time using the more useful versions of functions: Formulas and Graphs. And when a function is written as a formula with *x*'s and *y*'s, we usually assume that the *x*'s are the inputs and the *y*'s are the outputs.



❑ FORMULAS

EXAMPLE 1:

A. $y = 2x + 7$ is a function.

Given an input x, there is exactly one output y. For example, if $x = 4$, then $y = 15$ and nothing else. [You may recall that this would be called a *linear* function, and its graph is a straight line.]

B. $y = x^4$ is a function.

Given an input x, there is exactly one output y. For instance, $x = 3$ produces $y = 81$. If $x = 0$, then $y = 0$. And if $x = -3$, then the only output is 81. Remember that even though 81 is the output for two different inputs, the fact remains that $x = 3$ has only 81 for its output, and $x = -3$ also has only 81 for its output. So, for any value of x, the formula produces exactly one value of y. This is why it's a function.

C. $y = -3x^2 + 7x - 19$ is a function. [In fact, it's a *quadratic* function.]

D. $y = |x|$ is a function, called an *absolute value* function.

If $x = 7$, then $y = 7$ — exactly one output for the input of 7.
If $x = -7$, then $y = 7$ — exactly one output for the input of –7.

E. $x = |y|$ is not a function.

Why isn't it a function? We need to conjure up some input (an x) which has more than one output (y). Let's choose $x = 4$. This gives us the equation $4 = |y|$, and there are two solutions to this equation; namely, y could be 4, or y could be –4. We have more than one output for a single input. It follows that this formula does not represent a function.

F. $3x - 7y = 19$ is a function.

Here we could solve for y,

$$y = \frac{3}{7}x - \frac{19}{7},$$

and see that given an x, there's only one y for it, so this is a (linear) function.

G. $x^2 + y^2 = 4$ is not a function.

Let $x = 0$. Then $y^2 = 4 \Rightarrow y = \pm 2$. That is, an input of 0 results in two outputs, 2 and –2. Thus, this formula (which you might know is a circle) is not a function.

H. $y = \pm\sqrt{x}$ is <u>not</u> a function.

If we choose 81 as the input, we get two outputs: ± 9, violating the fundamental notion of a function.

I. $y = 5$ <u>is</u> a function.

Given any input (the x), there's only one output, namely 5. Therefore, this horizontal line is a function.

J. $x = 2$ is <u>not</u> a function.

In fact, this example probably holds the world's record for being a non-function. After all, given an input (the only choice being 2), there are an <u>infinite</u> number of outputs (y can be <u>anything</u>). We thus see that a vertical line is not a function.

x	y
2	5
2	0
2	–3

Homework

1. Consider the formula $y = x^3$. Recall that x is an input and y is an output. When a value of x has been assigned, there's only one y value. What does this mean?

2. Consider $y^2 = x$. When $x = 25$, y has two values, 5 and –5. What does this mean?

3. Consider $x = |y + 10|$.

   a. When $x = 20$, y has two values. Verify this fact.

   b. What does this mean?

4. Determine whether the formula is a function:

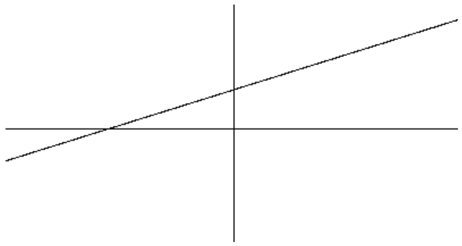
a. $y = -7x + 9$
b. $x = 2y + 5$
c. $y = x^5$
d. $y^2 = x$
e. $x^2 + y^2 = 14$
f. $y = |7x + 2|$
g. $x^2 - y^2 = 9$
h. $x = \pi$
i. $y = \sqrt{2}$
j. $y = \pm\sqrt{x-1}$
k. $y = \dfrac{x+1}{x-3}$
l. $y = \sqrt{2x^2 + x + 1}$
m. $x = |y + 5|$
n. $x^2 + y^2 = 1$
o. $y = \sqrt{1-x}$
p. $y = x^3 - x^2 + x$
q. $x - 5 = 0$
r. $y + 2\pi = 0$
s. $3x - 7y = 8$
t. $y = |10 - x|$
u. $x = |1 - y|$

☐ GRAPHS

Now it's time to determine whether or not a given graph is a function. We will assume that the standard *x*- and *y*-axes are used, and as before, we agree that *x* is the input and *y* is the output. Recalling that a function must produce exactly one output for each legal input, we look at the following three graphs.

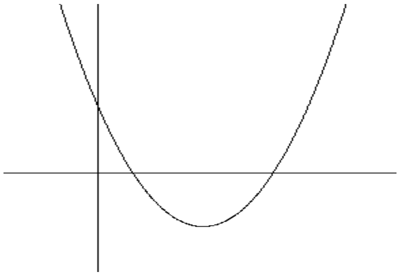
EXAMPLE 2:

A.



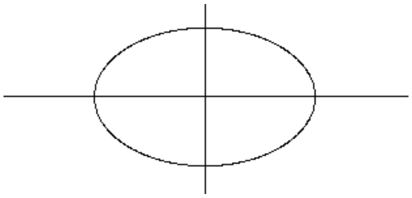
Choose any *x*-value on the *x*-axis — this is the input. Now go straight up or down until you get to the graph to find the *y*-value — this is the output. How many outputs did you get? You should have gotten exactly one output. Whatever legal x you use, there's exactly one y for it. The graph is a function.

B.



Pick any input on the x-axis. Go up or down until you hit the graph — the y-value is the output. Every legal value of x produces exactly one output. This graph is also a function.

C.



Choose a legal input (say, $x = 0$). Now go find the graph. This time we run into the graph twice, once going up and once going down. Our x has two different y's. That is, we have a legal input with more than one output. This graph is definitely not a function.

The following chart indicates other examples of graphs and their status as functions:

FUNCTIONS	NON-FUNCTIONS
single point	vertical line
horizontal line	circle
top-half of a circle	ellipse (oval)
Non-vertical line	right-half of a circle
parabola opening up	parabola opening to the left

Homework

5. T/F: No line is a function.

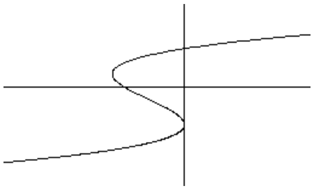
6. Which of the following are functions?

   a. horizontal line b. bottom half of a circle c. left half of a circle

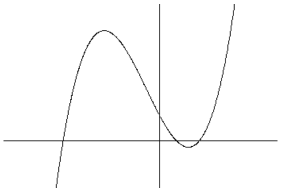
7. Explain why a graph consisting of a single point is a function.

8. Which of the following are functions?

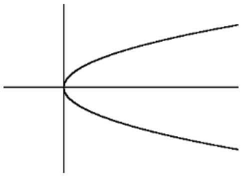
a.



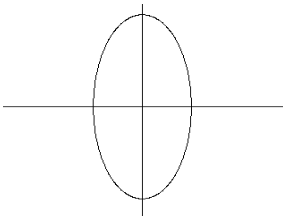
b.



c.



d.



Review Problems

9. Consider the formula $x^2 + y^2 = 1$. Explain why it's not a function.

10. Consider the formula $y = \pm\sqrt{x}$. If $x = 9$, what is y? Is this a function?

11. Is it a function?

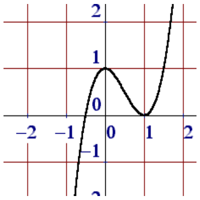
a. $x = |y + 1|$ b. $y = x^3$ c. $y = \pm\sqrt{x - 3}$

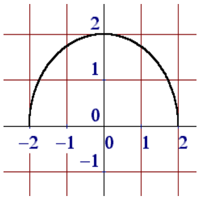
d. $x = 5$ e. $y = -\pi$ f. $x^2 = y^2$

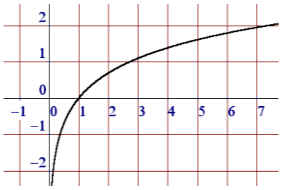
g. $7x - 9y = 10$ h. $y = |x + 5|$ i. $y = 7x^2 - \pi x^{10}$

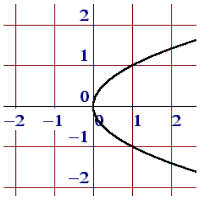
j. $x^2 + y^2 = 49$ k. $x = 0$ l. $y = 0$

12. Which are functions?









13. True/False:

a. $x = y^4$ is a function.

b. $y = |x - 2|$ is a function.

c. $y = \pm\sqrt{x - \pi}$ is a function.

d. $x = 9$ is a function.

e. $y = -\pi$ is a function.

f. $x^2 + y^2 = 2$ is a function.

g. Every line is a function.

h. Every semicircle is a function.

i. $x = |y - 1|$ is a function.

Solutions

1. It means that the formula represents a function, since each input has a unique output.

2. It means that the formula does <u>not</u> represent a function, since there's an input with more than one output.

3. a. If $x = 20$, then $20 = |y+10|$. This implies that

$$y + 10 = 20 \text{ or } y + 10 = -20, \text{ giving two solutions: } y = 10, -30.$$

b. One input (20) produced two outputs (10 and –30). Therefore, the formula does <u>not</u> represent a function.

4.
- a. Yes, given an input (x), there's only one output (y).
- b. Yes, solve for y and it's just like part a.
- c. Yes, one input produces exactly one output.
- d. No, an x-value of 4 produces two y-values, namely 2 and –2.
- e. No, an x-value of 0 produces two y-values.
- f. Yes
- g. No, if $x = 10$, then y has two values.
- h. No, for $x = \pi$, there are an infinite number of y's.
- i. Yes, given any input, the output must be $\sqrt{2}$, so there's a unique output for each input.
- j. No, if $x = 9$, then y has two values, 2 and –2.
- k. Yes, put in any legal value of x, and only one y-value will result.
- l. Yes, it may be complicated but only one y-value will appear for a given x-value.
- m. No, if $x = 2$, then y can be either –3 or –7.
- n. No, let $x = 0$ and see what you get.
- o. Yes, as long as an appropriate x is chosen, the positive square root produces exactly one answer.
- p. Yes q. No r. Yes s. Yes t. Yes u. No

5. F (actually, most lines are functions)

6. a. and b.

7. When you choose the only legal *x*, you're already at the *y*-value, and there's only one.

8. b. only

9. If $x = 0$, $y = \pm 1$; an input has produced two outputs, so it's not a function. Or, the graph is a circle, which is certainly not a function.

10. $y = \pm 3$; it is not a function, since a single input of 9 produced two outputs.

11. a. No b. Yes c. No d. No e. Yes f. No
g. Yes h. Yes i. Yes j. No k. No l. Yes

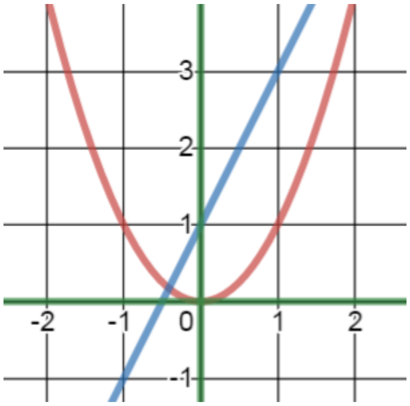
12. All but the 4th

13. a. F b. T c. F d. F e. T f. F
g. F h. F i. F

> “Ninety-nine percent of the failures come from people who have the habit of making excuses.”
>
> – George Washington

CH XX – FUNCTION NOTATION AND COMPOSITION

The notation "*y = whatever*" is a fine way to write a function, but it has a major flaw. Suppose we are talking about two functions in the same problem — for example, the line $y = 2x + 1$ and the parabola $y = x^2$. I now ask you what the *y*-value is when $x = 1$. Is the answer 3 (from the line), or is the answer 1 (from the parabola)? It depends on which formula you chose.



If $x = 1$, what does *y* equal? It depends on which graph you're looking at.

☐ *FUNCTION NOTATION*

This is unacceptable — we'd never be able to communicate this way. This is why we use a new notation for functions: We give each function a different name, in order to distinguish one from the other. We can name our line and parabola functions something like

$$L(x) = 2x+1 \quad \text{and} \quad P(x) = x^2$$

[The symbol $L(x)$ is read "*L* **of** *x*" or "*L* **at** *x*".]

The *L* and the *P* are the names of the functions, and the *x*'s in parentheses just make it clear which variable is the input. So, for example, in the function *L*, the input is *x*, and the output is $L(x)$ (that is, $2x + 1$).

Now, with this new notation, if I specify an input of $x = 1$, it's fair to ask you what $L(1)$ is, and you'll know for sure that it's 3, because I gave you both the input (the 1) and the function L. And by the same token, if I asked what $g(1)$ is, you'd be confident that it's 1. Get it??

The following are some of the math functions you will find in programming languages and spreadsheets:

```
abs
sqrt
sin
log
exp
mod
round
```

EXAMPLE 1: **Given the three functions**

$$\mathbf{f(x) = 3x - 10}$$
$$\mathbf{g(x) = x^2 + 1}$$
$$\mathbf{h(x) = \frac{1}{x}}$$

calculate each of the functional values:

A. $f(5) = 3(\mathbf{5}) - 10 = 15 - 10 = \mathbf{5}$

B. $g(5) = \mathbf{5}^2 + 1 = 25 + 1 = \mathbf{26}$

C. $h(5) = \mathbf{\frac{1}{5}}$

D. $f(10) = 3(\mathbf{10}) - 10 = 30 - 10 = \mathbf{20}$

E. $g(\sqrt{7}) = (\sqrt{\mathbf{7}})^2 + 1 = 7 + 1 = \mathbf{8}$

F. $h(0) = \frac{1}{\mathbf{0}}$, which is **undefined**

Homework

1. Let $f(x) = x^2 + 3x$

$g(x) = 5 - x$

$h(x) = \sqrt[3]{x+1}$

Calculate:

a. $f(7)$ b. $g(7)$ c. $f(0)$ d. $g(0)$
e. $f(10)$ f. $g(10)$ g. $f(-5)$ h. $g(-5)$
i. $h(0)$ j. $h(-28)$ k. $h(-1)$ l. $f(-1)$

☐ COMPUTER LANGUAGE FUNCTIONS

Let's look at how a computer language handles functions by looking at three functions: two that are built into the language, `abs` and `sqrt`, and one that we'll define ourselves.

You may have guessed that **`abs`** stands for absolute value. For example, the `abs` function produces the outputs for the given inputs:

`abs`(7) = 7 `abs`(–12) = 12 `abs`(0) = 0

As for the square root function, **`sqrt`**, a computer would calculate like this:

`sqrt`(81) = 9 `sqrt`(0) = 0 `sqrt`(–4) = *Error*

Our third example will be a *user-defined* function; this is a function that does not come pre-built into the language, but one that we create ourselves.

Notice that the square root of a negative number does not exist as a real number — hence the *Error* in the `sqrt` calculations above. So how can a programmer absolutely guarantee that her program will never accidentally try to take the square root of a negative number? Here's one way: Take the number's absolute value first, then feed that result (which can't be negative) into the square root function:

`sqrt(abs`(–9)) would produce an output of **3**.

Do you see why? Starting with the input, –9, first its absolute value is calculated, which results in the number 9. Second, the positive square root of the 9 is taken, with a final output of 3. Get it? What if we start with a positive number? No problem — for example,

`sqrt(abs(100))` would produce an output of **100**.

The absolute value of 100 is still 100, whose square root is 10. Whether the input is positive, zero, or negative, our new function will always be able to take a square root, with no possible Error messages. And this new function will even work perfectly if the input is 0.

Just for the heck of it, let's reverse the order of the functions which comprise our new user-defined function, and see how it handles an input of –25:

`abs(sqrt(–25)) = abs(`***Error!***`)`

It appears that the order in which you carry out your functions might make everything fall apart, thus defeating the purpose of a user-defined function.

And we can even name our user-defined function (the one that works, not the one that starts with `abs`), and have it stored in the computer language, available anytime we need it. For instance, we might call the new function `RealSqrt`, meaning that our new function will always result in a real number and never produce an Error message — a good thing for a programmer to have. Depending on the computer language, it might be as simple to create as this:

```
Function RealSqrt(x)
   RealSqrt(x) = sqrt(abs(x))
End Function
```

Our new function, `RealSqrt`, is actually *composed* of two functions, `abs` and `sqrt`, performed one after the other. As such, we say that our new function is the ***composition*** of the two functions. So, if you wanted to use the function with an input of –100, you could write

`RealSqrt(–100)`, and the output would be `10`.

❑ COMPOSITION OF FUNCTIONS

Here's the mathematical approach to the composition of functions. A function takes an input and produces exactly one output. Isn't it possible that this output might itself be the input to some other function? For example, suppose we have the two functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x - 10$$

Our starting point will be to use the input 7 and place it into the function f. This yields an output of **49**. Now, declare the 49 to be the input to the function g. The output at this point is **39**. Skipping the middle step, the original input of 7 produced — by performing f and then g — an output of **39**.

Another way to describe this "composing" of functions is the following: The starting number of 7 was put into the "squaring machine," creating the number 49. Then the 49 was put into the "subtract 10 machine," giving the final answer of **39**.

Since the function f was done first, and the function g was done second, and the starting number was 7, we can consider "g of f of 7," written

$$g\big(f(7)\big) = \mathbf{39}$$

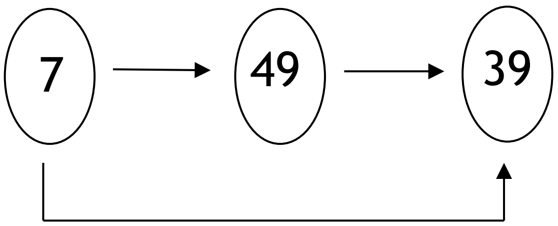
Showing all the details, we can write

$$g\big(f(7)\big) = g\big(7^2\big) = g(49) = 49 - 10 = \mathbf{39}$$

Using the output of one function as the input to a second function is called "composing" the functions. The function that takes the 7 directly to the 39 is called the "composition" of the two functions. It's a single function which has the same effect as the two original functions, one followed by the other.

In summary,

$f(x) = x^2 \qquad g(x) = x - 10$



The composition function takes 7 to 39 directly

f takes 7 to 49, and then g takes 49 to 39. We can write

$$g\big(f(7)\big) = 39$$

<u>EXAMPLE 2:</u> **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, calculate**

a. $f\big(g(-3)\big)$ **b.** $g\big(f(2)\big)$

<u>Solution:</u>

a. $f\big(g(-3)\big) = f\big(2(-3)+10\big) = f(4) = 4^3 - 4 = 64 - 4 = \boxed{60}$

b. $g\big(f(2)\big) = g\left(2^3 - 2\right) = g(6) = 2(6)+10 = \boxed{22}$

❑ *OFFICIAL NOTATION FOR THE COMPOSITION OF FUNCTIONS*

Calculating an expression of the form $f\big(g(x)\big)$ is important enough to warrant its own notation. We write

$$\boxed{(f \circ g)(x) = f\big(g(x)\big)}$$

and call $f \circ g$ the ***composition*** (or composite) of f and g. That is, given two functions f and g, we can create a third function called the composition of f and g, denoted by $f \circ g$, defined by the above equation. We can read $f\big(g(x)\big)$ as "f of g of x." Notice that the function g is done first, followed by f. Thus, function composition is done from right to left.

EXAMPLE 6: **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, find $(g \circ f)(x)$.**

Solution: By the definition of the composition of two functions,

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^3 - x) = 2(x^3 - x) + 10 =$$

$$\boxed{2x^3 - 2x + 10}$$

Function Composition is NOT Commutative

Remember the commutative property for multiplication? It says that if a and b are real numbers, then $ab = ba$. We'll now demonstrate that the composition of functions is a non-commutative operation. In other words, the order of the functions may make a difference in the final answer.

Let $f(x) = x^2$ and $g(x) = x + 1$. Let $x = 10$; we'll calculate

$$(f \circ g)(10) = f\big(g(10)\big) = f(10+1) = f(11) = 11^2 = \mathbf{121},$$

but

$$(g \circ f)(10) = g\big(f(10)\big) = g(10^2) = g(100) = 100 + 1 = \mathbf{101}.$$

Homework

2. Let $u(x) = x^2$ and $w(x) = 3x - 4$. Calculate:

a. $u(w(3))$ b. $w(u(3))$

3. Let $f(x) = \sqrt{x}$ and $h(x) = 3x$.

   a. Calculate $(f \circ h)(1)$

   b. Explain why $(f \circ h)(-3)$ does not exist (as a real number).

4. Let $f(x) = \sqrt{x}$ and $g(x) = x + 10$.

   a. Find $(f \circ g)(-1)$ b. $(f \circ g)(-10)$ c. $(f \circ g)(-11)$

5. If $g(x) = x^2 + 1$ and $f(x) = 5x - 9$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

6. If $f(x) = 3x$ and $g(x) = \sqrt{x+1}$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

7. If $g(x) = (x + 1)^2$ and $h(x) = \sqrt{x}$, calculate formulas for $(g \circ h)(x)$ and $(h \circ g)(x)$.

8. If $f(x) = 7x - 5$, calculate a formula for $(f \circ f)(x)$.

9. If $h(x) = x^2 + x - 1$, calculate a formula for $(h \circ h)(x)$.

10. Let $f(x) = \sqrt{x+4}$. Find a formula for $(f \circ f \circ f)(n)$.

    Now calculate $(f \circ f \circ f)(437)$.

❑ To ∞ and Beyond

A. Consider the following five functions:

$$f(x) = x^2 \qquad g(x) = x + 6 \qquad h(x) = \sqrt{x+17}$$

$$j = \left\{\left(\pi, \sqrt{2}\right), \left(\sqrt{2}, \pi\right)\right\} \qquad k = \{\,(a, \pi), (b, e)\,\}$$

Calculate: $(h \circ g \circ f \circ j \circ k)(a)$

B. Solve for n: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = 3$

C. Solve for n: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = w$

Solutions

1. a. $f(7) = \mathbf{7}^2 + 3(\mathbf{7}) = 49 + 21 = 70$ b. $g(7) = 5 - \mathbf{7} = -2$

c. 0 d. 5 e. 130 f. -5 g. 10

h. 10 i. 1 j. -3 k. 0 l. -2

2. a. $u(w(3)) = u(5) = 25$ b. 23

3. a. $f(h(1)) = f(3) = \sqrt{3}$ b. It would be $\sqrt{-9}$, not in $\mathbb{R}$

4. a. 3 b. 0 c. $\sqrt{-1}$, not in $\mathbb{R}$

5. $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 5(x^2+1) - 9 = 5x^2 - 4$

$(g \circ f)(x) = g(f(x)) = (5x-9)^2 + 1 = 25x^2 - 90x + 82$

6. $(f \circ g)(x) = 3\sqrt{x+1}$ $(g \circ f)(x) = \sqrt{3x+1}$

7. $(g \circ h)(x) = (\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$

$(h \circ g)(x) = \sqrt{(x+1)^2} = |x+1|$

8. $(f \circ f)(x) = f(f(x)) = f(7x-5) = 7(7x-5) - 5 = 49x - 40$

9. $x^4 + 2x^3 - x - 1$

10. $\sqrt{\sqrt{\sqrt{n+4}+4}+4}$

Evaluating the function at $n = 437$ gives a result of 3.

"One child, one teacher, one book, one pen can change the world."

\- Malala Yousafzai

CH NN – PIECEWISE (BRANCH) FUNCTIONS

A shipping company might offer a discount if the shipping weight exceeds a certain minimum. For example, the shipping fee might be \$3/lb for a shipment weighing less than 20 lbs, and \$2/lb for a weight of 20 lb or more.

Our federal income tax system uses a progressive tax; higher-income earners pay a higher *percentage* of their taxable income than lower-income earners. For instance (and this is an oversimplification), those who earn under \$50,000/yr might pay at a rate of 12% of their income, those who earn between \$50,000 and \$90,000 pay 15%, and those who earn more than \$90,000 pay 20%.

<u>EXAMPLE 1:</u> **For the shipping company in the Introduction,**

A. Find the cost to ship 12 lbs.

B. Find the cost to ship 35 lbs.

C. Express the shipping cost, S, as a branch function of the shipping weight, w.

Solution:

A. Since $w = 12$, which is less than 20, it costs \$3/lb, for a total of 12 lbs × \$3/lb = **\$36**.

B. 35 lbs at \$2/lb (because 35 lbs > 20 lbs) comes to **\$70**.

C. In English, we could say:

If $w < 20$, then $S = 3w$.

If $w \geq 20$, then $S = 2w$.

In our new notation, this is written

$$S = \begin{cases} 3w & \text{if } w < 20 \\ 2w & \text{if } w \geq 20 \end{cases}$$

EXAMPLE 2: **Find the piecewise (branch) function for the income tax in the Introduction, and then use the function to find the tax on an income of \$88,000.**

Solution: Here's the branch function, where I = income and T = tax:

$$T = \begin{cases} 12\% \times I & \text{if } I < 50{,}000 \\ 15\% \times I & \text{if } 50{,}000 \leq I \leq 90{,}000 \\ 20\% \times I & \text{if } I > 90{,}000 \end{cases}$$

To find the tax on an income of \$88,000, we find which branch of the tax function the number 88,000 lies in. It lies in the middle branch, so we use the formula 15% × I, giving a tax of

$$T = 15\% \times \mathrm{I} = 0.15(88{,}000) = \mathbf{\$13{,}200}$$

<u>EXAMPLE 3</u>: Let's look more closely at the absolute value function:

$$f(x) = |x|$$

We know, for example, that $f(9) = 9$, $f(-17) = 17$, and $f(0) = 0$. Here's one way we can define **absolute value,** using the **if/then** form:

If $x \geq 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

As a branch function, we write

$$\boxed{|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}}$$

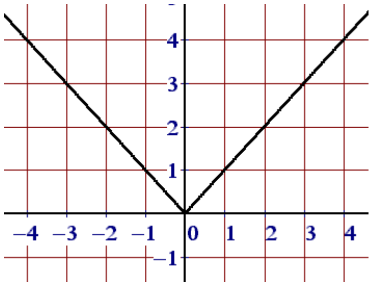
In programming, it might be written:

```
If x >= 0 Then
    Abs(x) = x
Else
    Abs(x) = -x
Endif
```

In summary, we call a function described in the three examples above a ***branch function*** (or piecewise function), because the formula we use to calculate the output depends upon which branch of the domain the input lies in.

For example, in the *absolute value* branch function, suppose $x = 9$. Since $9 \geq 0$, we use the formula in the top branch; so $|9| = 9$. If we take x to be 0, we are again in the top branch, and thus $|0| = 0$. But if we take $x = -17$, then x falls in the bottom branch, whose formula is $-x$. So $|-17| = -(-17) = 17$, just as it should be.

Note: The definition of absolute value we've just written tells us that the domain of *f* is ℝ, all real numbers. Here's how you can tell: Choose <u>any</u> real number, and then see that it <u>must</u> fall into exactly one of the two "if's"; that is, the real number you chose is either ≥ **0** or it's < **0**. All the bases are covered, so the absolute value function is defined for all real numbers.



EXAMPLE 4: **Consider the branch function defined by**

$$\boldsymbol{f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 2x & \text{if } x > 3 \end{cases}}$$

I. Calculate: **a.** $\boldsymbol{f(-8)}$ **b.** $\boldsymbol{f(0)}$ **c.** $\boldsymbol{f(3)}$

d. $\boldsymbol{f(200)}$ **e.** $\boldsymbol{f\left(\frac{\pi}{2}\right)}$

II. What is the domain of *f*?

III. Graph *f*.

IV. Analyze all the interesting limits.

Solution: To calculate function values, we first need to determine whether the input we're looking at falls in the first branch ($x \le 0$), in the second branch ($x > 3$), or possibly in no branch at all.

I.

a. Since $-8 \le 0$, $f(8) = 8^2 = \mathbf{64}$.

b. Also, $0 \le 0$, so $f(0) = 0^2 = \mathbf{0}$.

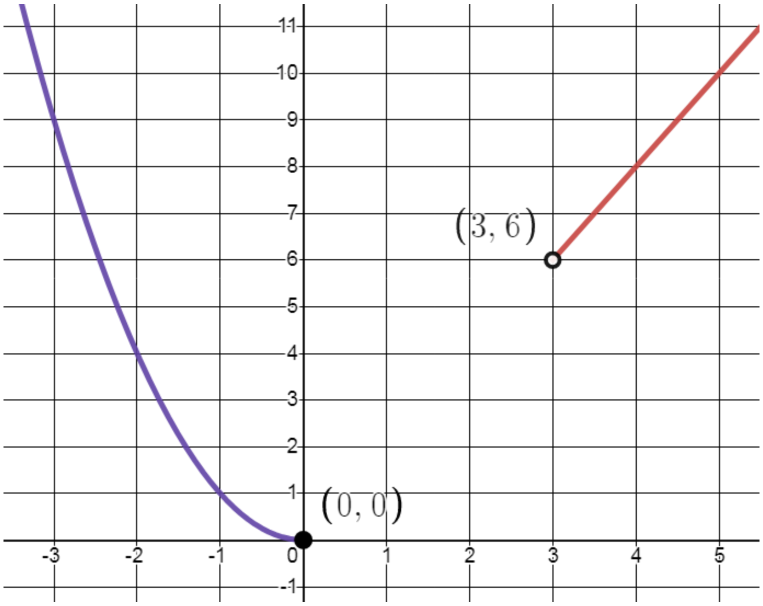
c. The value $x = 3$ is not in either branch. Thus, $f(3)$ is **undefined**.

d. Because $200 > 3$, $f(200) = 2(200) = \mathbf{400}$.

e. Note that $\frac{\pi}{2}$ lies between 0 and 3 — it falls through the cracks of the domain. Therefore, $f\left(\frac{\pi}{2}\right)$ is **undefined**.

II. The domain is explicitly stated in the definition of the branch function — namely all real numbers that are either ≤ 0 or > 3. That is, the domain is $\mathbf{(-\infty, 0] \cup (3, \infty)}$.

III. For $x \le 0$, the graph is a piece of the parabola $y = x^2$. For all $x > 3$, the graph is the straight line $y = 2x$. So the left piece of the branch function is a parabola, and the right piece is a line, and there's a giant gap between 0 and 3.



Note that there's a solid dot at the origin, because *f* is a parabola for any $x \le 0$. In other words, 0 is in the domain; thus the point (0, 0) is on the graph. On the other hand, the point (3, 6) is <u>not</u> on the graph, since 3 is not in the domain — hence the open dot.

IV. Now for some limits:

As $x \to 3$ (from the right), $f(x) \to 6$,

but x cannot approach 3 from the left.

As $x \to 0$ (from the left), $f(x) \to 0$,

yet x cannot approach 0 from the right.

As $x \to \infty$, $f(x) \to \infty$ (up the straight line).

As $x \to -\infty$, $f(x) \to \infty$ (up the parabola).

Final Comment: To say that $x \le 0$ means that $x \in (-\infty, 0]$. Also, if $x > 3$, then $x \in (3, \infty)$. Thus, an equivalent form of the original branch function f in this example is

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 0] \\ 2x & \text{if } x \in (3, \infty) \end{cases}$$

The notation $x \in A$ means that *x* is an **element** of A.

Also, while I use the word "if," some books use "for."

Homework

1. Let h be the function defined by $h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

   A. What is the domain of h?

   B. Find $h(13)$, $h(0)$, and $h(-\pi)$.

   C. Graph h.

2. Consider the branch function defined by $f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ 3x & \text{for } x > 2 \end{cases}$

   A. Calculate: a. $f(-8)$ b. $f(0)$ c. $f(2)$
   d. $f(200)$ e. $f(\pi/3)$

   B. What is the domain of f?

   C. Graph f.

3. Let g be the piecewise function defined by

$$g(x) = \begin{cases} x^3 & \text{if } x \in (-\infty, -2) \\ -x^2 & \text{if } x \in [-1, 1] \\ 2 & \text{if } x \in (1, \infty) \end{cases}$$

   A. Calculate: a. $g(1/2)$ b. $g(99)$ c. $g(-4)$ d. $g(-1.5)$
   e. $g(-5)$ f. $g(-2)$ g. $g(-1)$ h. $g(1)$

   B. What is the domain of g?

   C. Graph g.

Check out:

https://www.mathsisfun.com/sets/functions-piecewise.html

☐ TO ∞ AND BEYOND

Define the function f by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \pi & \text{if } x \text{ is rational} \end{cases}$

Find $(f \circ f)\left(\frac{2}{3}\right)$.

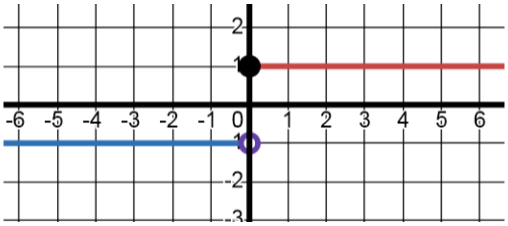
Solutions

---

1. A. The domain is $\mathbb{R}$, all real numbers, since every real number falls in exactly one of the two branches of the domain.

B. 1; 1; –1

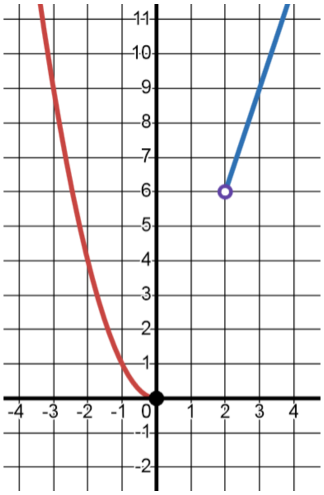
C.



2. A. a. 64 b. 0 c. Undefined d. 600 e. Undefined

B. $(-\infty, 0] \cup (2, \infty)$

C.



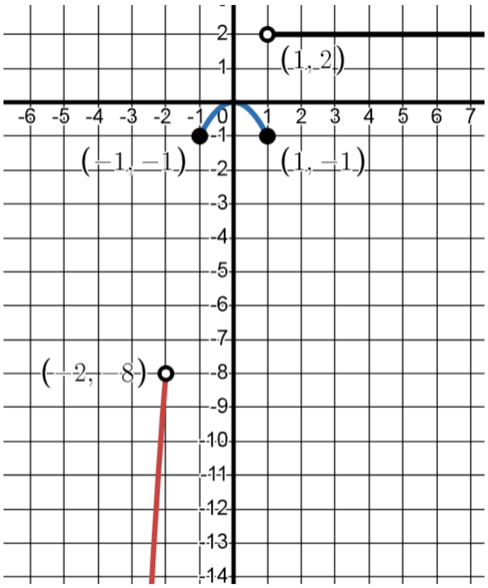
3. A. a. $-\frac{1}{4}$ b. 2 c. −64 d. −3.375
e. −125 f. Undefined g. −1 h. −1

B. It's just the union of the three intervals given in the function:

$$(-\infty, -2) \cup [-1, 1] \cup (1, \infty)$$

But note that the last two intervals can be combined into $[-1, \infty)$.

C.



CH XX – FUNCTIONS, REWRITING FORMULAS

The formula for the area of a square is $A = s^2$, where s is the length of a side. Since A is based on s, we can say that A is a function of s, insofar as A depends on s. We can even use the terminology: s is the input and A is the output.

The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height. Here we say that V is a function of both r and h.

To express X as a function of b means to write an equation where X is on the left side of the equation, and the right side consists of some formula containing only the variable b (and perhaps some constants).

EXAMPLE 1: **Express the side of a square as a function of its area.**

Solution: The formula $A = s^2$ expresses A as a function of s. To express the side s as a function of the area A, all we need to do is solve the formula for s:

$$\boxed{s = \sqrt{A}}$$

Now s is a function of A.

NOTES: First, this may seem like a trivial problem, but it's crucial when using computers. If you want to find the side of a square given the area, then you must do the algebra to solve for s — the computer cannot do it for you.

The second note is that technically, solving $A = s^2$ for s should result in $s = \pm\sqrt{A}$. But the side of a square can never be negative, so in this context it's perfectly fine to drop the "±" sign and consider only the non-negative square root of A.

EXAMPLE 2: **Express the area of a circle as a function of its circumference.**

Solution: Normally, we think of the area of a circle as a function of its radius, namely $A = \pi r^2$, and we also consider the circumference as a function of the radius, as in $C = 2\pi r$.

To express the area, A, as a function of the circumference, C, we need to write a formula with A on the left side and some formula with C (but not r) on the right side. To accomplish this, we will remove the r from the formula $A = \pi r^2$ and insert the C from the formula $C = 2\pi r$. It's a substitution trick.

$$C = 2\pi r \qquad \text{(usual circumference formula)}$$

$$\Rightarrow \quad r = \frac{C}{2\pi} \qquad \text{(divide each side by } 2\pi\text{)}$$

Now substitute this version of r into the area formula:

$$A = \pi r^2 \qquad \text{(usual area formula)}$$

$$\Rightarrow \quad A = \pi\left(\frac{C}{2\pi}\right)^2 \qquad \text{(since } r = \frac{C}{2\pi} \text{ from above)}$$

$$\Rightarrow \quad A = \pi\left(\frac{C^2}{4\pi^2}\right) \qquad \text{(square the fraction)}$$

$$\Rightarrow \quad \boxed{A = \frac{C^2}{4\pi}} \qquad \text{(cross-cancel)}$$

EXAMPLE 3: **Express the surface area of a sphere as a function of its diameter.**

Solution: The standard formula for the surface area of a sphere is $S = 4\pi r^2$, a formula in which the surface area S is written as a function of the radius r. To express the area as a function of the diameter, we need to get rid of the variable r and replace it with d.

From the formula $d = 2r$, we solve for r: $r = \frac{d}{2}$. Thus,

$$S = \underbrace{4\pi r^2 = 4\pi\left(\frac{d}{2}\right)^2}_{\text{since } r = \frac{d}{2}} = 4\pi\left(\frac{d^2}{4}\right) = \pi d^2$$

Hence, $\boxed{S = \pi d^2}$

To help you absorb the concept of "something as a *function* of something else," let's summarize the formulas we've just seen.

$A = s^2$	A is a function of s
$V = \pi r^2 h$	V is a function of r and h
$s = \sqrt{A}$	s is a function of A
$r = \frac{C}{2\pi}$	r is a function of C
$A = \pi r^2$	A is a function of r
$A = \frac{C^2}{4\pi}$	A is a function of C
$S = 4\pi r^2$	S is a function of r
$d = 2r$	d is a function of r
$r = \frac{d}{2}$	r is a function of d
$S = \pi d^2$	S is a function of d

Homework

1. Express the area of a square as a function of its side.

2. Express the area of a square as a function of its perimeter.

3. Express the perimeter of a square as a function of its area.

4. Express the leg of a right triangle as a function of its other leg and hypotenuse.

5. Express the length of a rectangle as a function of its width and area.

6. Express the circumference of a circle as a function of its diameter.

7. Express the diameter of a circle as a function of its circumference.

8. Express the area of a circle as a function of its diameter.

9. Express the circumference of a circle as a function of its area.

10. Express the volume of a cylinder as a function of its height and radius.

11. Express the radius of a cylinder as a function of its volume and height.

12. Express the radius of a sphere as a function of its surface area.

13. Express the volume of a sphere as a function of its diameter.
    ($V = \frac{4}{3}\pi r^3$)

14. Express the radius of a sphere as a function of its volume.

15. Express the volume of a sphere as a function of its surface area.

16. Express the surface area of a sphere as a function of its volume.

Review Problems

17. a. Express the volume of a cube as a function of its surface area.

    b. Express the surface area of a cube as a function of its volume.

18. In the formula for the mean, $\bar{x} = \frac{x_1 + x_2}{2}$, express x_1 as a function of x_2 and $\bar{x}$.

19. True/False:

    a. $A = \pi r^2$ gives the radius of a circle as a function of its area.

    b. $S = \pi d^2$ gives the surface area of a sphere as a function of its diameter.

    c. If $G = \sqrt{2}\,h^5$, it's possible to write h as a function of G.

Solutions

1. $A = s^2$

2. $A = s^2$; but $P = 4s \Rightarrow s = \frac{P}{4}$; substituting into the area formula:

$A = \left(\frac{P}{4}\right)^2 \Rightarrow A = \frac{P^2}{16}$.

3. $P = 4s$; but $A = s^2 \Rightarrow s = \sqrt{A}$; substituting gives $P = 4\sqrt{A}$.

4. Since, $(\text{leg1})^2 + (\text{leg2})^2 = h^2$, it follows that $\text{leg1} = \sqrt{h^2 - (\text{leg2})^2}$.

5. $l = \frac{A}{w}$

6. $C = 2\pi r$; $d = 2r \Rightarrow r = \frac{d}{2}$. Therefore, $C = 2\pi\left(\frac{d}{2}\right) \Rightarrow C = \pi d$.

7. $d = \frac{C}{\pi}$

8. $A = \pi r^2$; $r = \frac{d}{2}$. Therefore, $A = \pi\left(\frac{d}{2}\right)^2 \Rightarrow A = \frac{\pi d^2}{4}$.

9. $C = 2\pi r$; $A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$. Thus, $C = 2\pi\sqrt{\frac{A}{\pi}}$.

10. $V = \pi r^2 h$ **11**. $r = \sqrt{\frac{V}{\pi h}}$ **12**. $r = \sqrt{\frac{S}{4\pi}}$

13. $V = \frac{4}{3}\pi r^3$; $r = \frac{d}{2}$. Therefore, $V = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \Rightarrow V = \frac{\pi d^3}{6}$.

14. $r = \sqrt[3]{\frac{3V}{4\pi}}$ **15**. $V = \frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{3/2}$ **16**. $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

17. a. $V = \left(\frac{A}{6}\right)^{3/2}$ b. $A = 6V^{2/3}$

18. Solving the mean formula for x_1 gives $x_1 = 2\bar{x} - x_2$.

19. a. F b. T c. T

> **"On matters of style, swim with the current; on matters of principle, stand like a rock."**
>
> Thomas Jefferson

CH XX – FUNCTIONS, TABLES AND MAPPINGS

The notion of a *function* is much more abstract than most of the algebra concepts you've seen thus far, so we'll start with three specific non-math examples.

❑ *THE MEANING OF A FUNCTION VIA TABLES*

First Example: A football game is divided into four quarters. The following table shows each quarter together with the total number of points scored during that quarter.

Quarter	Number of Points Scored
1st	21
2nd	9
3rd	0
4th	25
Inputs	Outputs

This table is a ***function***, and here's why:

First, we have a set of **inputs**, the four quarters:

1st 2nd 3rd 4th

Second, we have a set of **outputs**, the total points scored in each quarter:

21 9 0 25

Third, there's a definite connection, or correspondence, between the inputs and the outputs. For example, the input "2nd" is associated with the output "9." What about the input "4th"? The output is "25." Similarly, "21" is the output for the input "1st," while "3rd" produces the output "0."

Fourth, and most importantly, notice that each input has exactly one output. For instance, the input "2nd" has exactly one output, namely "9." There can be no argument that the output is 9 — just look at the table. Also, the input "4th" clearly has the output "25" and nothing else. This is the essence of a function, so our table of football quarters and points scored is a function.

Second Example: So, what isn't a function? Consider the following table:

City	Major League Baseball Teams
San Francisco	Giants
Atlanta	Braves
New York	Yankees
New York	Mets
Inputs	Outputs

The input San Francisco has exactly one output, Giants, because the city of San Francisco is home to exactly one major league baseball team. The input Atlanta also has exactly one output associated with it. So far, so good – in terms of being a function.

But check out New York: It has two baseball teams, the Yankees and the Mets. We thus have an input (New York) with more than one output (Yankees and Mets). This kills the function concept, and we conclude that our table of cities and baseball teams is not a function.

To summarize this example, for something to be a function, there must be exactly one output for each input; this example has an input with two outputs. That's why the city/team pairings do NOT constitute a function.

Third Example:

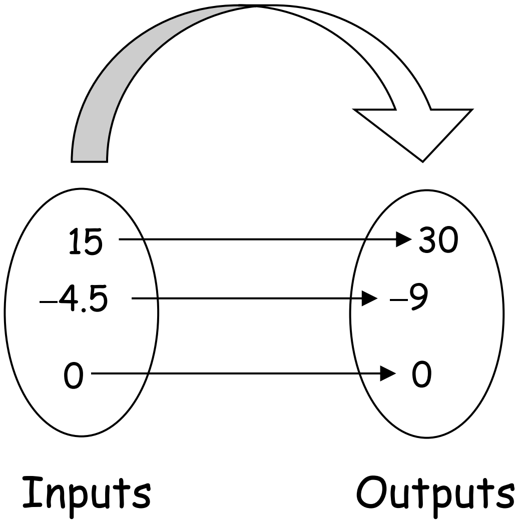
City	Population
Elmville	35,000
Gomerville	47,500
Moeville	35,000
Inputs	Outputs

Is this table a function? Does each input have exactly one output? YES, and I'll prove it to you. The input Elmville has one output (35,000), the input Gomerville has one output (47,500), and the input Moeville has one output (35,000). Does it matter that Elmville and Moeville have the same population? Not at all — it's just a coincidence. The fact is, every city has exactly one population associated with it; that is, each input has exactly one output. We must conclude that the population table is a function — although this function somehow differs from the function in the first example.

❑ *MATHEMATICAL EXAMPLES OF FUNCTIONS VIA MAPPINGS*

EXAMPLE 1: **Analyze the *doubling* function.**

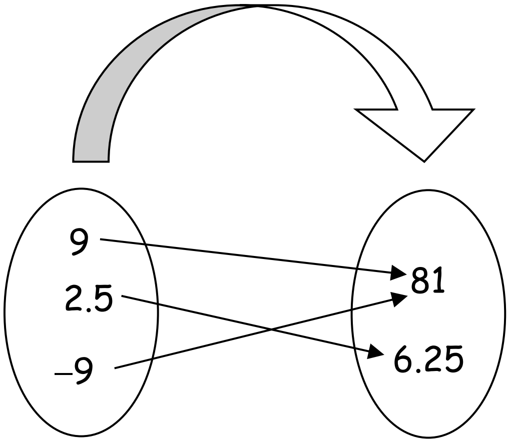
Solution: Consider the ***doubling function***. This function takes an input, and produces an output that is double the input. For example, if the input is 15, then the output is double 15, or 30. That is, this function takes an input of 15 and produces an output of 30. In the diagram below we see three inputs (15, –4.5, and 0) producing three outputs (30, –9, and 0).



The
DOUBLING
Function

EXAMPLE 2:

A. Look at the ***squaring*** function.



The
SQUARING
Function

Why is this a function? Because, given an input, there is exactly one output for it. Notice that even though both 9 and –9 produce the same output of 81, it is still the case that 9 squared is 81, and only 81 — and that –9 squared is 81, and only 81. Each input has exactly one output: *squaring* is a function.

B. Consider the ***square root*** function, $\sqrt{x}$. An input of 49 produces an output of 7, and an input of 2 produces an output of $\sqrt{2}$. Recall that the formula represents the non-negative square root only; so, for instance $\sqrt{100} = 10$, and <u>not</u> –10. [If you want both square roots, write: $\pm\sqrt{100}$.]

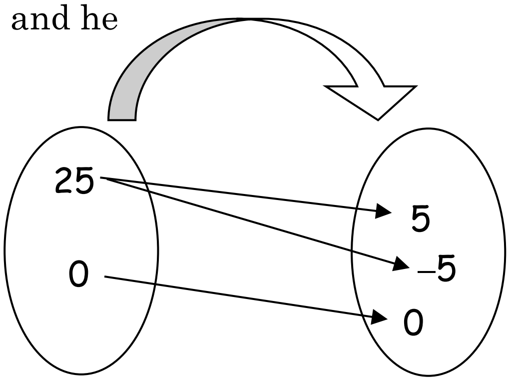
What inputs are allowed in this function? Since negative numbers don't have real-number square roots, all inputs must be at least zero, i.e., zero or bigger.

To reiterate, this is a function because given any legal input ($x \ge 0$), we obtain exactly <u>one</u> output.

C. Now for the ***reciprocal*** function; that is, $\frac{1}{x}$. An input of 7 produces $\frac{1}{7}$, an input of $-\frac{2}{3}$ produces $-\frac{3}{2}$, and an input of 0 produces nothing at all, since 0 has no reciprocal. That is, 0 is not a legal input to the reciprocal function. What can x be in this function? Well, the only number we can't divide by is 0, so x can be any real number except 0.

Starting with any legal input ($x \ne 0$), we obtain exactly one output — we conclude that the reciprocal is a function.

E. Now consider the operation "**take <u>both</u> square roots**": $\pm\sqrt{x}$. Is this a function? Suppose we take the input $x = 25$. Ask a person what the output is, and he might say 5, since 5 is indeed a square root of 25. But ask another person, and she might say –5, since –5 is also a square root of 25. Who's right? They're both right! There's nothing illegal or immoral about our formula. The fact is, it's simply <u>not</u> a function — there's an input that produces more than one output.



Homework

1. Construct a table of inputs and outputs that represents a function.

2. Construct a table of inputs and outputs that does not represent a function.

3. Construct a mapping (the circles and arrows) demonstrating "absolute value." Is it a function? Explain.

4. Construct a mapping that represents a non-function. Explain precisely why it's not a function.

5. Using the functions in Example 2, calculate the output for the three given inputs:

   a. The *squaring* function: $\frac{2}{3}$ -15 π
   Is there any number that cannot be used as an input?

   b. The (non-negative) *square root* function: 144 1 5
   Is there any number that cannot be used as an input?

   c. The *reciprocal* function: $\frac{1}{4}$ -3 1
   Is there any number that cannot be used as an input?

6. True/False:

   a. In the tripling function, if the input is 12, the output is 36.

   b. In the cubing function, if the output is 125, the input is 5.

Solutions

1. Be sure that every input has exactly one output.

2. Be sure that at least one input has more than one output.

3. I'd like to see what you came up with.

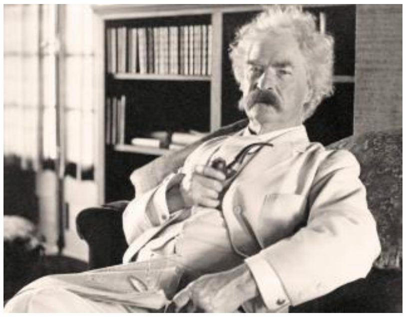
4. Be sure you have at least one input with 2 or more arrows diverging from it.

5.

a.	4/9	225	π^2	Every number can be used as an input.
b.	12	1	$\sqrt{5}$	No negative number can be square-rooted.
c.	4	$-\frac{1}{3}$	1	The only number without a reciprocal is 0.

6. a. T b. T

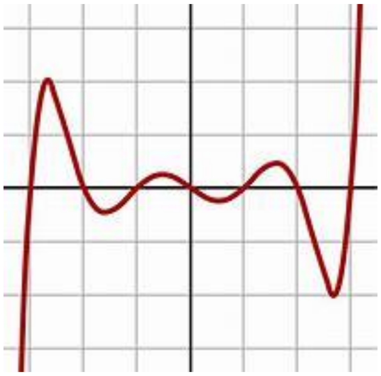
"The secret of getting ahead is getting started. The secret of getting started is breaking your complex overwhelming



tasks into small manageable tasks, and then starting on the first one."

– Mark Twain

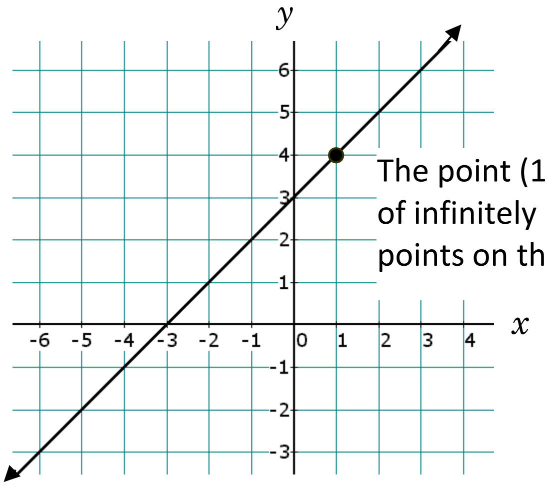
CH *N* – FROM GRAPH TO EQUATION



In this chapter, I'm going to give you a graph, but only a small portion of it, generally around the origin. We will first analyze points we can actually see (for instance, I will ask what *y*-value is associated with a certain *x*-value). Then I will ask you what *y*-value is associated with an *x*-value that you can't even see on the graph! How will you do this? You will use the given graph to create an algebraic formula to describe the graph, and then use that formula to predict *y*-values from any *x*-value I care to give you.

☐ *LINES*

EXAMPLE 1: **Consider the following graph:**



a. **Calculate the y-value when $x = 100$.**

b. **Calculate the y-value when $x = 2.7$.**

c. **Calculate the x-value when $y = 1000$.**

Solution: We certainly don't want to extend (draw) the graph all the way to $x = 100$. So our approach, described in the Introduction, will be two-fold:

1) Determine the equation of the given graph by analyzing the relationship between x and y.
2) Use that equation to deduce the y-value for the given x-value.

The first step will be to create an x-y table by reading the coordinates off the graph. For example, on the given graph, we see that when $x = 1$, $y = 4$; thus the point (1, 4) is on the graph.

x	–4	–3	–2	–1	0	1	2	3
y	–1	0	1	2	3	4	5	6

Remember that the arrowheads on the graph indicate that the graph (the line) goes forever in both directions. It then follows that x can be any number at all. For example, although it's not easy to see from the graph, if we choose x to be 1.5, it can be guessed that $y = 4.5$. Similarly, we could let $x = \pi$ and then see that y would be a little bigger than 6.

Now let's examine the equation of the line. Looking at the x-values and y-values in the table, we might see that the y-value is always **3 more than** the x-value. We are

Guessing certainly isn't very accurate, and that's even when we have a graph to look at. It would be even more difficult to find points on the line if we had to extend the graph ourselves. This is why it's so very important, when possible, to create an *equation* from the given graph. Assuming that we end up with the right equation, we are guaranteed exact y-values for any given x-value.

therefore led to the formula

$$y = x + 3$$

Check this formula against all the x-y pairs in the table and you will see that it really works. Now we can answer the three questions:

a. If $x = 100$, then $y = x + 3 = \mathbf{100} + 3 = \mathbf{103}$.

b. If $x = 2.7$, then $y = x + 3 = \mathbf{2.7} + 3 = \mathbf{5.7}$.

c. Be careful here; note that y is given and we want to find x.

$y = x + 3$ (the formula for the line)

$\Rightarrow$ $1000 = x + 3$ (let $y = 1000$)

$\Rightarrow$ $1000 - 3 = x + 3 - 3$ (subtract 3 from each side)

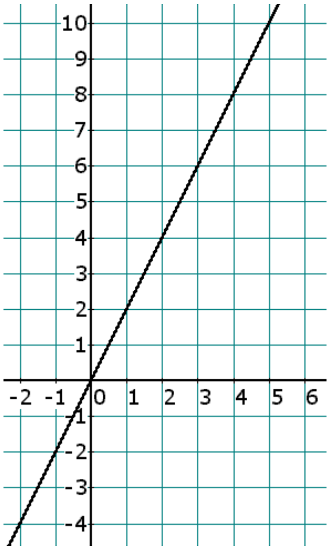
$\Rightarrow$ $\mathbf{997} = x$ (solve for x)

In summary, the following three points lie on the line:

(100, 103) (2.7, 5.7) (997, 1000)

Homework

1. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 3$, $y =$ ___.

b. When $x = 0$, $y =$ ___.

c. When $x = -4$, $y =$ ___.

d. When $x = 278$, $y =$ ___.

e. When $x = -900$, $y =$ ___.

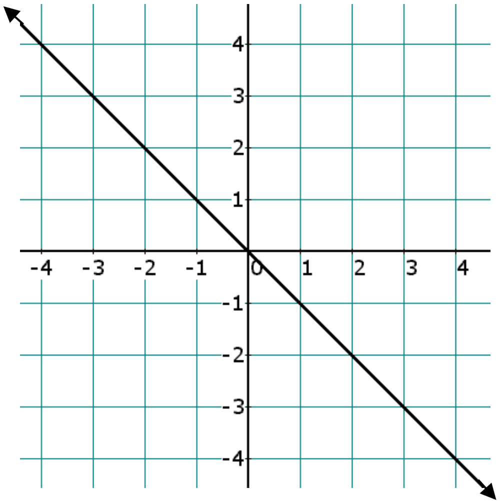
f. When $x = 18\pi$, $y =$ ___.

g. When $y = 10$, $x =$ ___.

h. When $y = 250$, $x =$ ___.

i. When $y = 22\pi$, $x =$ ___.

EXAMPLE 2: **Consider the following graph:**



a. **Calculate the y-value when $x = 1200$.**

b. **Calculate the y-value when $x = -123$.**

c. **Calculate the x-value when $y = -25\pi$.**

Solution: Again, the points in which we're interested are not visible on the graph, so let's create a table based on the graph, and then use that table to construct a formula.

x	−4	−3	−2	−1	0	1	2	3
y	4	3	2	1	0	−1	−2	−3

We see that the y-value is simply the ***opposite*** of the x-value:

$$y = -x$$

Notice that when $x = 0$, $y = -0 = 0$ (which yields the origin); therefore, 0 has an opposite, namely 0. We conclude that every number has an opposite. We can now answer the three questions:

a. If $x = 1200$, then $y = -x =$ **-1200**.

b. If $x = -123$, then $y = -x = -(-123) =$ **123**.

c. As in the previous example, the y-value is given and we're asked for the x-value. So let's begin with the formula, plug in the given value of y, and then solve for x:

$$y = -x \qquad \text{(the formula for the line)}$$

$$\Rightarrow \quad -25\pi = -x \qquad \text{(let } y = -25\pi\text{)}$$

$$\Rightarrow \quad 25\pi = x \qquad \text{(divide or multiply each side by } -1\text{)}$$

Therefore, the following three points lie on the line:

$$\boxed{(1200, -1200) \quad (-123, 123) \quad (25\pi, -25\pi)}$$

Homework

2. The graph of $y = -x$ lies within Quadrants ___ and ___ and at the origin.

3. What is the **opposite** of each number?

a. 17 b. 0 c. -3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.

b. The opposite of 0 is ____.

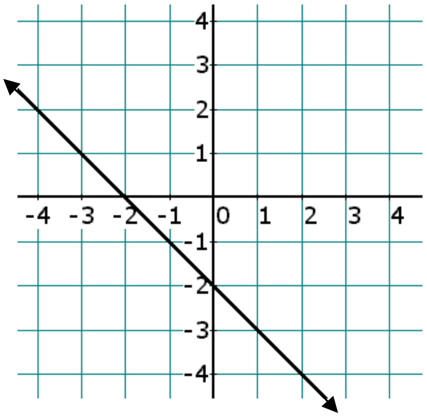
c. The opposite of a negative number is always _____.

d. The opposite of a positive number is always _____.

5. Consider the formula $y = -x + 5$. Calculate the y-value for the given x-value:

a. 8   b. 99   c. –10   d. –5   e. 0   f. π g. $-\pi$

6. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 2$, $y =$ ___.

b. When $x = 0$, $y =$ ___.

c. When $x = -4$, $y =$ ___.

d. When $x = 25$, $y =$ ___.

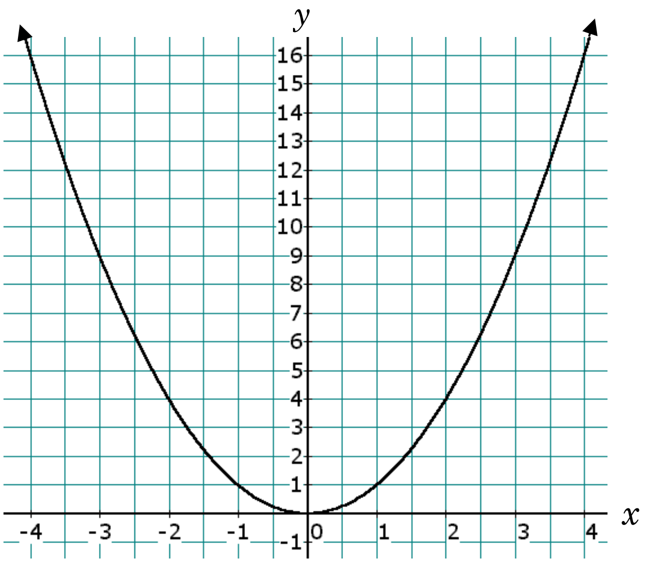
e. When $x = -200$, $y =$ ___.

f. When $y = 32$, $x =$ ___.

g. When $y = -300$, $x =$ ___.

☐ *Beyond Lines*

EXAMPLE 3: **Consider the following graph:**



a. **Calculate the y-value when $x = 1{,}000$.**

b. **Calculate the y-value when $x = 1.2$.**

c. **Calculate the x-values when $y = 625$.**

Solution: This graph is much different from the previous graph. It passes through the origin, and the rest of the graph resides only in Quadrants I and II. Let's try to read off some ordered pairs from the graph:

x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Now, how is the y-value related to the x-value? When $x = 0$, $y = 0$; this is rather useless, as is the pair (1, 1). But look at the ordered pairs (2, 4), (3, 9), and (4, 16). There's something going on here — it appears that the y-value is simply the square of the x-value:

$$y = x^2$$

Lest we jump to conclusions, let's double-check the rest of the pairs in the table. For example, is the square of –4 equal to 16? Yes. Is the square of –1 equal to 1? Yes. Is the square of 0 equal to 0? Yes. Our formula $y = x^2$ is looking better and better. Notice that even though x can be any number at all, the y-value is never negative; the y-values start at 0 and go toward positive infinity. Now to answer the original questions:

a. If $x = 1{,}000$, then $y = x^2 = 1{,}000^2 = \mathbf{1{,}000{,}000}$.

b. If $x = 1.2$, then $y = x^2 = 1.2^2 = \mathbf{1.44}$.

c. Notice that this part of the problem asks for the x-values, plural. I guess this means that a y-value of 625 might have two x-values associated with it. This seems reasonable from the table — if, for instance, given a y-value of 9, we notice that both 3 and –3 are x-values that yield a y-value of 9. In other words, if $y = 9$, then $x = 3$ or -3. So, what two numbers, when squared, would yield a result of 625? Well, $25^2 = 625$, and of course $(-25)^2 = 625$.

In short, the following four ordered pairs lie on the curve:

(1000, 1000000)	(1.2, 1.44)	(25, 625)	(–25, 625)

Homework

7. The graph of $y = x^2$ lies within Quadrants ___ and ___ and at the origin.

8. Using the formula $y = x^2$, answer each question:

   a. If $x = 50$, then $y =$ ___.

   b. If $x = -25$, then $y =$ ___.

   c. If $x = 0$, then $y =$ ___.

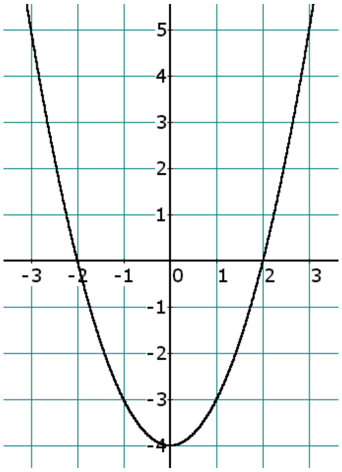
   d. If $y = 49$, then $x =$ ___ or ___.

   e. If $y = 144$, then $x =$ ___ or ___.

   f. If $y = -9$, then $x =$ ___.

9. If $y = 2x^2 - 3x - 1$, then what is y if $x = -5$?
   Recall: The Order of Operations requires that exponents be done before multiplication, which is to be done before any adding or subtracting.

10. Consider the graph at the right:



a. Fill in the following x-y table.

x	–3	–2	–1	0	1	2	3
y							

b. Find a formula that relates x and y. Hint: There's an x^2 in the formula.

c. When $x = 100$, $y =$ ___.

d. When $x = –20$, $y =$ ___.

e. What is the lowest point on the graph?

f. How many y-values are associated with the x-value 1,000?

g. How many x-values are associated with the y-value 1,000?

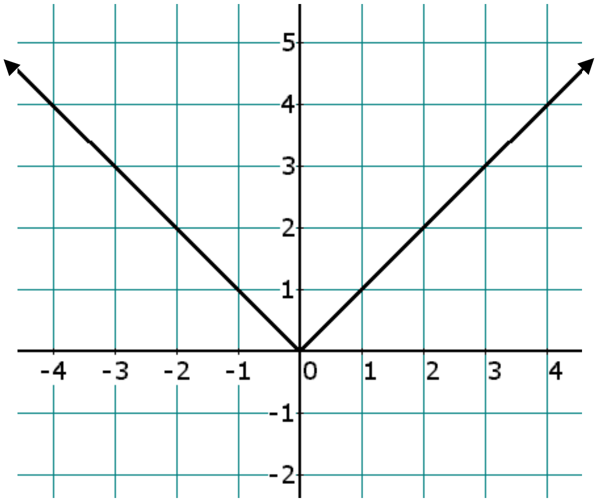
h. If $y = –3$, then $x =$ ___ or ___.

i. If $y = 96$, then $x =$ ___ or ___.

j. If $y = 221$, then $x =$ ___ or ___.

k. If $y = –6$, then $x =$ ___.

EXAMPLE 4: **Consider the following graph:**



a. Calculate the y-value when $x = 239$.

b. Calculate the y-value when $x = -777$.

c. Calculate the x-values when $y = 250$.

Solution: It's not a line and it's not curvy, so what is it? As usual, we begin with a table whose entries are read from the graph:

x	–4	–3	–2	–1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4

How is the y-value connected to the x-value? Well, when x is 0 or positive ($x \geq 0$), the y-value is the same as the x-value. But when x is negative ($x < 0$), the y-value is the opposite of the x-value.

Though the graph is straightforward (it's just the shape of the letter "V"), there's no simple, recognizable formula for the relationship between x and y, or is there? You might have learned about the notion of absolute value; we say that

y is the ***absolute value*** of x, and we write $y = |x|$.

Therefore, the vertical bar symbol around the x means "compute the absolute value of x," which means:

1. If $x \geq 0$, then $|x| = x$.
   The absolute value of a positive number or 0 is the number itself.

2. If $x < 0$, then $|x| = -x$.
   The absolute value of a negative number is the opposite of the number (thus making it positive).

We should now be ready to answer the following questions:

a. For $x = 239$, $y = |x| = |239| = \mathbf{239}$.

b. For $x = -777$, $y = |x| = |-777| = \mathbf{777}$.

c. We are asked for the x-values (plural) when $y = 250$. In other words, we have to solve the equation $\mathbf{250 = |x|}$, which asks: What numbers have an absolute value of 250? Well, since $|250| = 250$ and $|-250| = 250$, the two x-values are **250** and **–250**. We conclude that the following four points lie on the graph:

(239, 239) (–777, 777) (250, 250) (–250, 250)

Be sure you can visualize the location of these four points on the graph.

Homework

11. The graph of $y = |x|$ lies within Quadrants ___ and ___ and at the origin.

12. Find the **absolute value** of each number:

a. 72 b. –99 c. 0 d. π e. $-\pi$ f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17-7|$ b. $|3-25|$ c. $|2(3)-6(1)|$ d. $|2\pi+3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

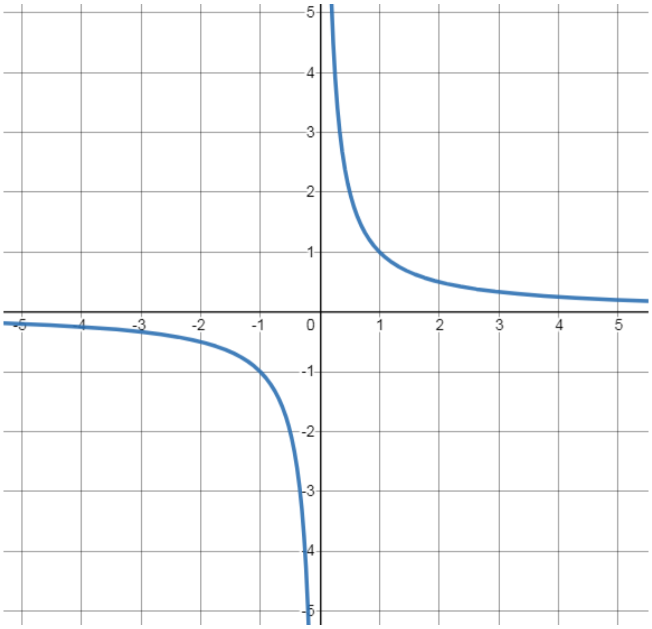
c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

EXAMPLE 5: **Consider the following graph:**



a. **Calculate the y-value when $x = 50$.**

b. **Calculate the y-value when $x = -1/10$.**

c. **Calculate the y-value when $x = 0$.**

Solution: We have another curvy graph, but this one consists of two distinct pieces. The most important characteristic of this graph is that x is *never* 0; that is, the graph never touches the y-axis (even though it does get infinitely close!).

As in the two previous examples, the questions asked of us cannot be answered by looking at the graph — we must find a formula using the points we can see on the graph, and then use that formula to predict the y-values for the given x-values. To find some points on this graph, you'll have to trust me to a certain extent, because it's not easy to read fractional numbers from such a rough picture. See if you can agree with the following claims:

When $x = 1$, it's pretty clear that $y = 1$. When $x = 2$, y looks like it's about 1/2 (or 0.5). And trust me, when $x = 3$, $y = 1/3$, and when $x = 4$, $y = 1/4$.

Now let $x = 1/2$; do you see that $y = 2$? How about when $x = 1/3$, then $y = 3$? This quite nicely analyzes the first quadrant. Note again that x cannot be 0, because if you start at the origin (where $x = 0$), you can go up or down the y-axis as far as you'd like and never run into the graph. It's time for a summary:

x	1/3	1/2	1	2	3	4
y	3	2	1	1/2	1/3	1/4

What is going on here? Using the point (3, 1/3) as a guide, we conjecture that the y-value is found by dividing 1 by the x-value; in other words, flip over the x-value to get the y-value:

$$y = \frac{1}{x}$$

Let's check our formula for $x = 1$ and $x = 1/3$:

$$x = 1 \quad \Rightarrow \quad y = \frac{1}{x} = \frac{1}{1} = 1 \ ✓$$

$$x = \frac{1}{3} \quad \Rightarrow \quad y = \frac{1}{x} = \frac{1}{\frac{1}{3}} = \frac{1}{1} \times \frac{3}{1} = 3 \ ✓$$

Now for a check of a negative x-value. Consider $x = -3$; the formula gives

$y = \frac{1}{-3} = -\frac{1}{3}$, yielding the point (–3, –1/3),

which seems very reasonable from the graph. We are now convinced that we have the right formula: $y = \frac{1}{x}$. We call this the ***reciprocal*** formula, and we'll use it to answer the original questions:

a. If $x = 50$, then $y = \frac{1}{x} = \mathbf{\frac{1}{50}}$, or **0.02**.

b. If $x = -1/10$, then $y = \frac{1}{x} = \frac{1}{-\frac{1}{10}} = \frac{1}{1} \times -\frac{10}{1} = \mathbf{-10}$.

c. If $x = 0$, then $y = \frac{1}{x} = \frac{1}{0}$, which is **undefined.** Notice that this is perfectly consistent with the earlier observation that x can never be 0 on the graph, and we see here that y can never be 0 in the formula, either.

Homework

15. The graph of $y = \frac{1}{x}$ lies entirely within Quadrants ___ and ___.

16. As x grows larger and larger, y is always (positive, negative), but getting (larger, smaller).

17. Using the formula $y = \frac{1}{x}$, answer each question:

a. If $x = 14$, then $y =$ ____.

b. If $x = \frac{2}{3}$, then $y =$ ____.

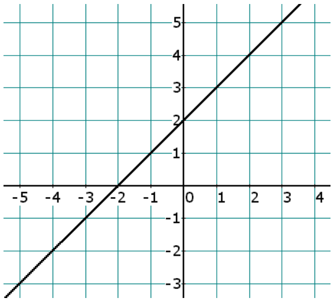
c. If $x = -99$, then $y =$ ____.

d. If $x = -\frac{5}{4}$, then $y =$ ____.

e. If $x = 0$, then $y =$ ____.

Review Problems

18. Consider the following graph:



a. When $x = 3$, $y =$ _____.

b. When $x = 99$, $y =$ _____.

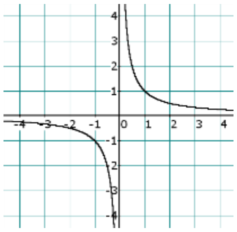
c. When $x = -45$, $y =$ _____.

d. When $y = 132$, $x =$ _____.

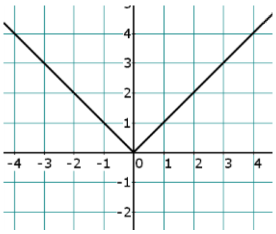
e. When $y = -33$, $x =$ _____.

19. What is the equation for each graph?

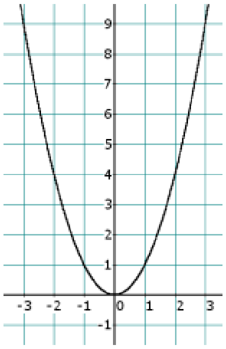
a.



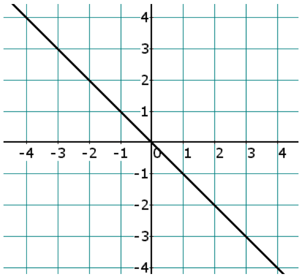
b.



c.



d.



20. Let $y = x^2$.

a. If $x = 17$, $y =$ _____

b. If $x = -13$, $y =$ _____

c. If $y = 16$, $x =$ _____

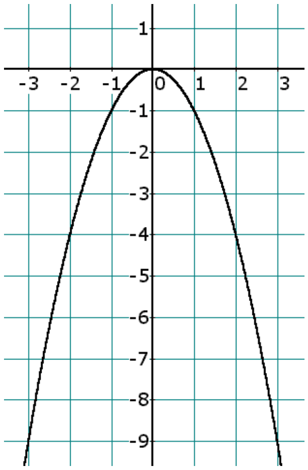
d. If $y = 0$, $x =$ _____

e. If $y = -9$, $x =$ _____

f. If $x = \pi$, $y =$ _____

g. Find the points on the graph where the two coordinates match.

21. Consider the graph:



a. If $x = 0$, then $y =$ _____

b. If $x = 10$, then $y =$ _____

c. If $x = -12$, then $y =$ _____

d. If $y = -225$, then $x =$ _____

e. If $y = 9$, then $x =$ _____

22. Evaluate: a. $|17|$ b. $|-\pi|$ c. $|0|$

23. Evaluate: $\dfrac{|-7|+|7|}{\frac{1}{2}+\frac{1}{3}}$

24. If $y = \dfrac{1}{x}$, and if $x = \dfrac{1}{10}$, what is y?

Solutions

1. a. 6 b. 0 c. -8 d. 556 e. -1800
f. 36π g. 5 h. 125 i. 11π

2. II and IV

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative

17

5. a. –3 b. –94 c. 15 d. 10
e. 5 f. $-\pi + 5$ g. $\pi + 5$

6. a. –4 b. –2 c. 2 d. –27
e. 198 f. –34 g. 298

7. I and II

8. a. 2,500 b. 625 c. 0 d. 7, –7 e. 12, –12
f. No answer (no real number squared could be negative)

9. 64

10. a. 5 0 –3 –4 –3 0 5 b. $y = x^2 - 4$ c. 9,996
d. 396 e. (0, –4) f. 1 g. 2 h. –1 1 i. 10 –10
j. 15 –15 k. No answer

11. I and II

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17, –17 e. 0 f. No answer

15. I and III

16. positive; smaller

17. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

18. a. 5 b. 101 c. –43 d. 130 e. –35

From Graph to Equation

19. a. $y = \frac{1}{x}$ b. $y = |x|$ c. $y = x^2$ e. $y = -x$

20. a. 289 b. 169 c. ±4 d. 0 e. Does not exist
f. π^2 g. Hint: There are two such points.

21. a. 0 b. –100 c. –144 d. ±15 e. Does not exist

22. a. 17 b. π c. 0

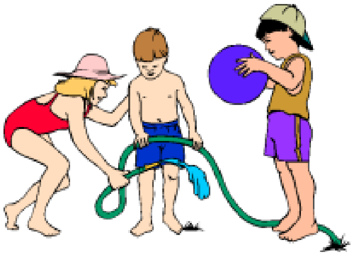
23. $\frac{84}{5}$ **24**. 10

“The greatest use of life is to spend it for something that will outlast it.”

William James (the father of modern psychology)

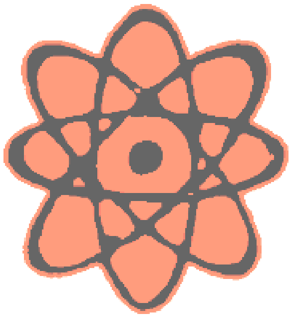
CH NN –THE GROWTH AND DECAY FORMULA

Many things grow and shrink (decay) at a rate based on how much stuff there is at the moment.



For instance, the number of births in a **population** is based on the number of people living at the time (since the more people there are, the more new people there will be).

Another example is **compound interest**: As money accumulates in the account (due to earned interest), the more interest the account will earn; that is, the interest will earn interest.



In science, we can observe the **radioactive decay** of an element like uranium. As the uranium decays, less of it will decay because as time goes on, there's less of it left to decay.

☐ THE GROWTH AND DECAY FORMULA

Here's a formula to help you predict how much of something there will be in the future. Among hundreds of applications, it works for business depreciation, the spread of disease, and the half-life of a drug:

Let

A_0 = starting amount (read: "A sub zero" or "A naught")
A = ending amount
e = a constant whose value is **approximately 2.718**.
k = growth or decay rate (expressed as a decimal)
t = time

Then

$$\boxed{A = A_0 e^{kt}}$$

This formula assumes **continuous** growth or decay, which means at "every moment in time." Don't be dismayed if that is hard to fathom.

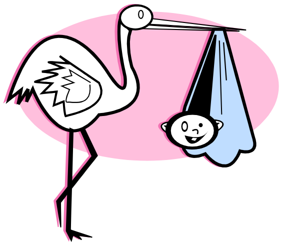
Get your calculator out. Here's a quick review of the "exponent" button. To calculate $(3.2)^{4.15}$, try either

3.2 $\boxed{y^x}$ 4.15 $\boxed{=}$ OR 3.2 $\boxed{\wedge}$ 4.15 $\boxed{=}$

The answer is about 124.845. [Your calculator may use $\boxed{\text{enter}}$ instead of the equals sign.]

The number ***e*** used in the growth formula is one of the most important numbers in math, statistics, science, engineering, and business. Like the number π, this real number contains an infinite number of digits which never form a repeating pattern (e and π and $\sqrt{2}$ are called *irrational* numbers).

EXAMPLE 1: **Assuming an initial population of 1,506 and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**



Solution: Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1,506; so $A_0 = 1{,}506$.

The growth rate is 25%; thus $k = 0.25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula (using 2.718 for e):

$$A = 1{,}506(2.718)^{(0.25)(10)}$$

$$\Rightarrow \quad A = 1{,}506(2.718)^{2.5}$$

$$\Rightarrow \quad A = 1{,}506(12.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 18{,}343}$$

EXAMPLE 2: **Assuming a starting investment of $2,401 and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**

Solution: The formula we used for population growth works equally well for compound interest.

$$A = A_0 e^{kt}$$

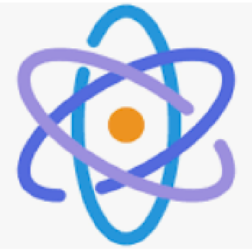
$$\Rightarrow \quad A = 2{,}401(2.718)^{(0.13)(11)}$$

$$\Rightarrow \quad A = 2{,}401(2.718)^{1.43}$$

$$\Rightarrow \quad A = 2{,}401(4.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = \$10{,}036}$$

EXAMPLE 3: **Starting with 624 grams of uranium and assuming an annual decay rate of 5%, compute the number of grams of uranium remaining after 9 years.**



Solution: We use the same growth/decay formula; but since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of **–5%** (that's negative 5 percent) in our formula:

$$A = A_0e^{kt}$$

$$\Rightarrow \quad A = 624(2.718)^{(-0.05)(9)}$$

$$\Rightarrow \quad A = 624(2.718)^{-0.45}$$

$$\Rightarrow \quad A = 624(0.64) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 399 \text{ grams}}$$

Homework

Note: The result of raising 2.718 to the exponent is assumed to be rounded to 2 digits, as in the examples.

1. Starting with 178 grams of uranium and assuming an annual decay rate of 17%, predict the number of grams remaining after 18 years.

2. Assuming an initial population of 3,902 and a growth rate of 14% per year, determine the population in 10 years.
3. Assuming an initial investment of $1,299 and an annual interest rate of 14% compounded continuously, predict the value of the investment in 4 years.
4. Assuming an initial investment of $9,574 and an annual interest rate of 11% compounded continuously, compute the value of the investment in 2 years.
5. Starting with 416 grams of thorium and assuming an annual decay rate of 3%, determine the number of grams remaining after 26 years.
6. Assuming an initial population of 2,586 and a growth rate of 16% per year, predict the population in 4 years.
7. Assuming an initial population of 1,422 and a growth rate of 12% per year, compute the population in 2 years.
8. Assuming an initial investment of $6,897 and an annual interest rate of 14% compounded continuously, compute the value of the investment in 6 years.
9. Starting with 215 grams of plutonium and assuming an annual decay rate of 15%, calculate the number of grams remaining after 26 years.
10. Assuming an initial population of 2,159 and a growth rate of 6% per year, calculate the population in 11 years.

[As mentioned on the second page of this Chapter, technically speaking, the growth/decay formula we've been using applies only when the growth or decay is *continuous*, which means the growth or decay occurs at every single moment. This may not strictly be the case in all situations (for example, in the births of people), but let's not worry about it in this chapter. Let's just use the formula to solve the problems.]

Solutions

1. 9 g
2. 15,803
3. $2,273
4. $11,968
5. 191 g
6. 4,913
7. 1,806
8. $16,001
9. 4 g
10. 4,167

> “Education is not the
> filling of a pail,
> but the lighting of a fire.”
>
> – *William Butler Yeats*

CH XX – THE HYPERBOLA

The hyperbola is the basis for a navigation system called LORAN, (Long Range Navigation.) The hyperbola is the shape of the orbits of certain comets, and is also used to design telescopes and the cooling towers at nuclear power plants.

☐ THE MAIN EXAMPLE

Graph the hyperbola $x^2 - y^2 = 9$.

First, let's find the **domain**. We solve for y, and then analyze all the legal values of x:

$$x^2 - y^2 = 9 \Rightarrow -y^2 = -x^2 + 9 \Rightarrow y^2 = x^2 - 9 \Rightarrow y = \pm\sqrt{x^2 - 9}$$

Because of the square root, y is a real number only when the radicand, $x^2 - 9$, is greater than or equal to zero: $x^2 - 9 \ge 0$. You can solve this inequality by the Boundary Point Method. When you do, you will conclude that the domain of the graph is **$(-\infty, -3] \cup [3, \infty)$**. By the way, can you explain why the hyperbola is <u>not</u> a function?

Second, we'll check out **intercepts**. Since 0 is not in the domain, there cannot possibly be any y-intercepts. [Indeed, letting $x = 0$ gives $0^2 - y^2 = 9 \Rightarrow y^2 = -9 \Rightarrow y = \pm\sqrt{-9}$, not members of $\mathbb{R}$.] Thus, there are no y-intercepts. As for x-intercepts, let $y = 0$; this gives $x^2 - 0^2 = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Therefore, there are two x-intercepts: **(3, 0)** and **(–3, 0)**.

Third, let's tackle **symmetry**. Replace x with $-x$: $(-x)^2 - y^2 = 9$, which is equivalent to $x^2 - y^2 = 9$, the original equation. Thus, the graph has y-axis symmetry. Similarly, you can show that the graph has x-axis symmetry. In fact, the graph of the hyperbola also has origin symmetry.

Fourth, we analyze **asymptotes**. Since there are no fractions in the hyperbola formula, it appears that the graph has no vertical asymptotes. To find any other possible asymptotes, we'll let $x \to \infty$ and see if y approaches anything interesting.

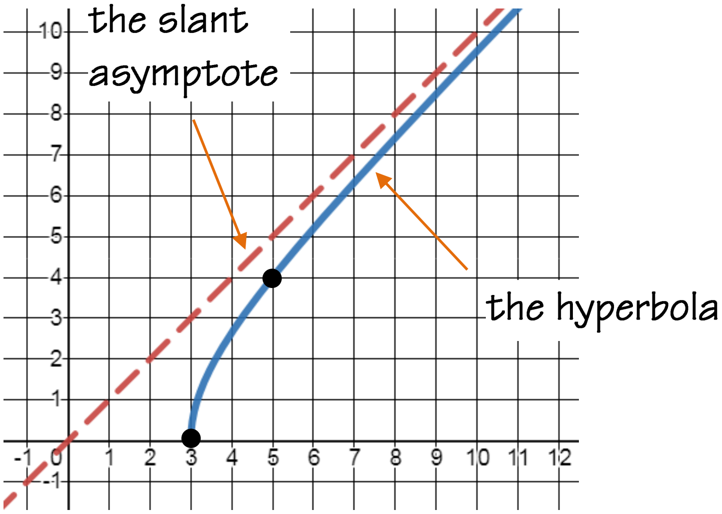
Let's construct a little x-y chart, and then choose really big values for x. Use your calculator to confirm these results.

x	y
5	± 4
10	± 9.5394
100	± 99.5499
1,000	± 999.9955
10,000	± 9,999.9996

What's going on here? Let's look just at the positive y-values. On the surface, it certainly appears that as x grows larger and larger, the y-values grow larger and larger. But look more carefully: The y-value is actually approaching the x-value. That is, as $\boldsymbol{x \to \infty, y \to x}$. We can write this limit as

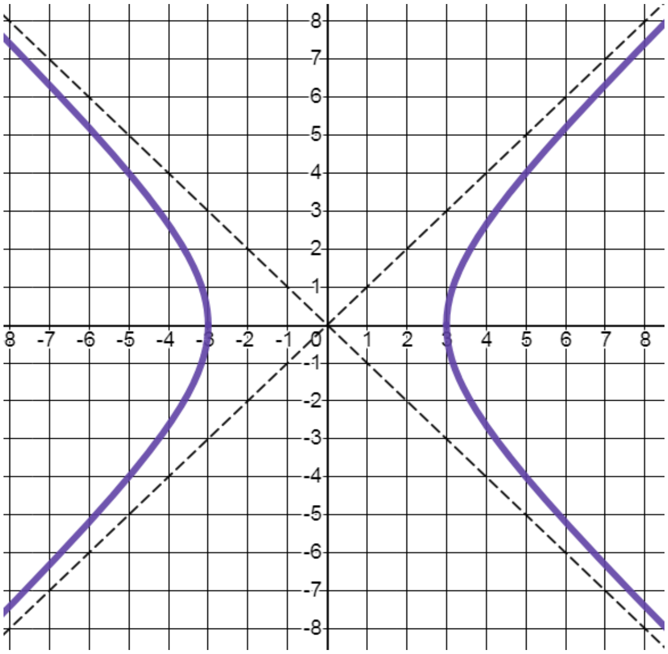
$$\lim_{x\to\infty} \sqrt{x^2 - 9} = x$$

This means that as x grows larger and larger, the curve is approaching the line $y = x$ (a 45° line passing through the origin). Since this line is neither vertical nor horizontal, we need a new adjective. We call this kind of asymptote a ***slant***, or ***oblique***, asymptote.



The analysis just completed gives a picture of the hyperbola in the first quadrant. It has an *x*-intercept at (3, 0), and a slant asymptote of $y = x$. Also, from the chart above, the hyperbola passes through the point (5, 4).

Since the hyperbola has *y*-axis symmetry, another piece of the hyperbola lies in the second quadrant. But the pieces of the hyperbola in Quadrants I and II have symmetric pieces in Quadrants III and IV, due to the *x*-axis symmetry of the hyperbola.



We're now ready to graph the entire hyperbola. Note that the dashed lines $y = x$ and $y = -x$ are the slant asymptotes of the hyperbola, and are graphed just for reference — they are <u>not</u> part of the hyperbola. Finally, the graph should convince you of the fact deduced earlier that the domain is $(-\infty, -3] \cup [3, \infty)$. In addition, the graph shows us that the **range** of the hyperbola is $\mathbb{R}$.

Homework

1. Regarding the main example above, $x^2 - y^2 = 9$, explain, using the formula, why it isn't a line, a parabola, a circle, or an ellipse.

2. Find the slant asymptotes for each hyperbola:

   a. $x^2 - y^2 = 25$ b. $x^2 - 9y^2 = 9$

   c. $36x^2 - y^2 = 36$ d. $9x^2 - 4y^2 = 36$

3. Find the domain and range of each hyperbola in the previous problem.

4. Perform a complete analysis (including a graph) on the hyperbola $x^2 - y^2 = 49$.

❑ *ANOTHER EXAMPLE*

Graph the hyperbola $y^2 - 4x^2 = 25$.

First we solve for y — this will allow us to find the domain and the slant asymptotes.

$$y^2 - 4x^2 = 25 \Rightarrow y^2 = 25 + 4x^2 \Rightarrow y = \pm\sqrt{25+4x^2}$$

Look at the radicand. No matter what kind of number x is, its square, x^2, is non-negative, which implies that $4x^2$ is non-negative, which implies that $25 + 4x^2$ is at least 25. So surely the radicand can never be negative. Thus, the **domain** of the hyperbola is $\mathbb{R}$.

Now it's time for the **intercepts**. Let $x = 0$; then $y^2 = 25$, or $y = \pm 5$. So we have a couple of y-intercepts: **(0, 5) and (0, –5)**. Now let $y = 0$; then $-4x^2 = 25 \Rightarrow x^2 = -\frac{25}{4} \Rightarrow$ no real solution for x. Therefore, there are no x-intercepts.

I'll leave it to you to verify that the hyperbola possesses all three types of **symmetry**.

To find the slant **asymptotes**, we refer to the form of the equation calculated above,

$$y = \pm\sqrt{25+4x^2}$$

Use your calculator to confirm the following ordered pairs (using just the positive square root.):

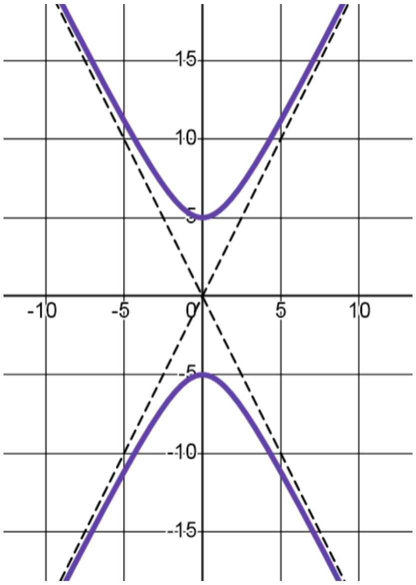
(10, 20.616) (100, 200.062) (1000, 2000.006)

As x grows larger and larger, the y-values appear to be growing closer and closer to whatever <u>twice x</u> is. That is, as $\boldsymbol{x \to \infty, y \to 2x}$. We write

$$\lim_{x\to\infty} \sqrt{25+4x^2} = 2x$$

Thus, one of the slant asymptotes is the line $\boldsymbol{y = 2x}$.

The graph goes through the intercept (0, 5) and then approaches the slant asymptote $y = 2x$. The x-axis and y-axis symmetries then give us the rest of the hyperbola in the other three quadrants. As before, the asymptotes (dashed) are graphed for reference.



Last, we discuss the **range** of the hyperbola. The y-values appear to be numbers that are either less than or equal to –5, OR greater than or equal to 5. That is, the range of the hyperbola is $\{y \in \mathbb{R} \mid y \le -5 \text{ or } y \ge 5\}$, which is $\boldsymbol{(-\infty, -5] \cup [5, \infty)}$.

Apollonius (Greek: 262 BC – 190 BC) gave the hyperbola its present name. See how the graph of the hyperbola seems to "fly away" in all directions? Look up the word "hyperbol<u>e</u>" in a dictionary (a big book that predates Google) and see if its meaning has any connection to the graph of a hyperbola.

Homework

5. Find the slant asymptotes for each hyperbola:

   a. $y^2 - x^2 = 144$ b. $y^2 - 16x^2 = 16$

   c. $9y^2 - x^2 = 9$ d. $25y^2 - 9x^2 = 225$

6. Find the domain and range of each hyperbola in the previous problem.

   Perform a complete analysis (including a graph) on each hyperbola:

7. $y^2 - x^2 = 25$

8. $y^2 - 9x^2 = 16$

9. $16y^2 - 9x^2 = 144$

10. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

11. Classify each of the following as a line, parabola, circle, ellipse, or hyperbola:

    a. $7x^2 - \pi x + 2y + 9 = 0$ b. $83x + 99y = 34$

    c. $7x^2 + 2y^2 - x = 10$ d. $y^2 + 7y - x^2 - 2x - 100 = 0$

    e. $7x + 99 = 400$ f. $32 - 9y + \pi = 0$

    g. $\sqrt{2}\,x^2 + \sqrt{2}\,y^2 - \sqrt{e}\,x + \sqrt[3]{\pi}\,y - 10^6 = 0$

Review Problems

12. Classify as parabola, circle, ellipse, hyperbola, or none of them:

a. $x^2 + y^2 = -1$ b. $y^2 - x^2 = 5$ c. $2x^2 + 4y^2 = 7$

d. $x^2 - y = 3$ e. $2x^2 - \pi y^2 = \sqrt{2}$ f. $3y^2 = x^2 + 5$

13. a. Find the intercepts of $3x^2 - 2y^2 = 2$.

b. Find the intercepts of $10y^2 - 30x^2 = 3$.

c. Find the domain of $x^2 - 14y^2 = 144$.

d. Find the range of $9x^2 - 36y^2 = 41$.

e. Find the asymptotes of $49x^2 - y^2 = 10$.

f. Find the asymptotes of $36y^2 - 4x^2 = 1$.

g. Find all symmetries of $4x^2 - 9y^2 = 36$.

14. Find the intercepts, domain, range, asymptotes, and symmetry of the hyperbola $x^2 - y^2 = 4$. Graph the hyperbola.

15. Find the asymptotes and the range of $y^2 - 9x^2 = 16$. Sketch the graph.

16. True/False:

a. Some of the hyperbolas in this chapter are functions.

b. The domain of the hyperbola is $x^2 - y^2 = 9$ is $[-3, 3]$.

c. Hyperbolas in this chapter always possess all three symmetries.

d. The slant asymptotes for $16x^2 - 9y^2 = 1$ are $y = \pm\frac{4}{3}x$.

e. There are no slant asymptotes for $25x^2 + 49y^2 = 1$.

f. The range of a hyperbola may consist of the union of two intervals.

g. Some of the hyperbolas in this chapter pass through the origin.

❑ *To ∞ and Beyond*

Find the **center** of the hyperbola

$$25x^2 - 9y^2 + 50x + 90y - 425 = 0$$

Hint: We note that the *origin* is the center of each hyperbola in this chapter.

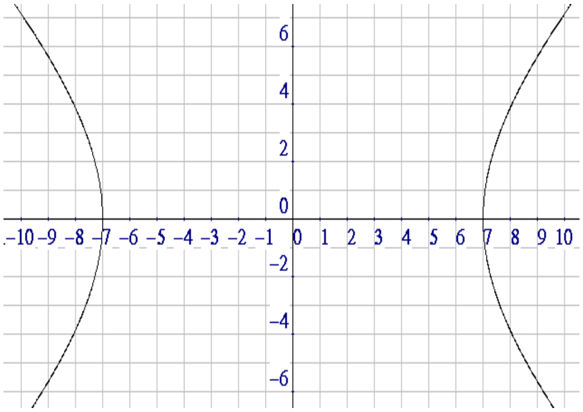
Solutions

1. Any letter squared ⇒ it's not a line
Both letters squared ⇒ not a parabola
Different coefficients on the squares ⇒ not a circle
Opposite signs on the coefficients on the squares ⇒ not an ellipse

2. a. $y = \pm x$ b. $y = \pm\frac{1}{3}x$ c. $y = \pm 6x$ d. $y = \pm\frac{3}{2}x$

3. a. $(-\infty, -5] \cup [5, \infty)$; $\mathbb{R}$ b. $(-\infty, -3] \cup [3, \infty)$; $\mathbb{R}$
c. $(-\infty, -1] \cup [1, \infty)$; $\mathbb{R}$ d. $(-\infty, -2] \cup [2, \infty)$; $\mathbb{R}$

4.



Domain = (–∞, –7] ∪ [7, ∞)

Range = ℝ Not a function

No *y*-intercepts

x-intercepts: (–7, 0) and (7, 0)

Symmetry: all three

$\lim_{x\to\infty}\sqrt{x^2-49} = x$

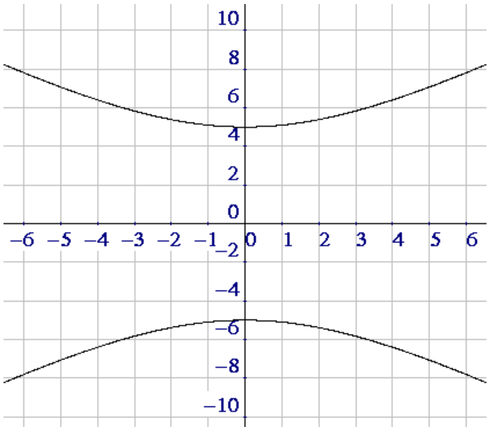
Slant asymptotes: $y = x$ and $y = -x$

5. a. $y = \pm x$ b. $y = \pm 4x$ c. $y = \pm\frac{1}{3}x$ d. $y = \pm\frac{3}{5}x$

6. a. ℝ; (–∞, –12] ∪ [12, ∞) b. ℝ; (–∞, –4] ∪ [4, ∞)

c. ℝ; (–∞, –1] ∪ [1, ∞) d. ℝ; (–∞, –3] ∪ [3, ∞)

7.

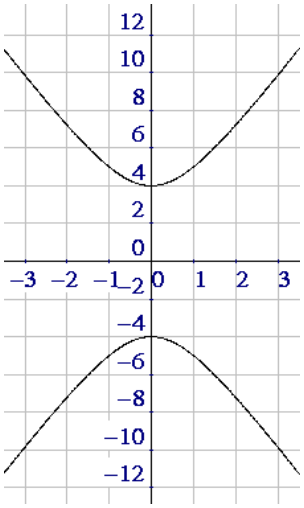


Domain = ℝ Range = (–∞ –5] ∪ [5, ∞) Not a function

No *x*-intercept; *y*-intercepts: (0, 5) and (0, –5)

Symmetry: all three Asymptotes: $y = x$ and $y = -x$

8.



Domain = $\mathbb{R}$

Range = $(-\infty, -4] \cup [4, \infty)$

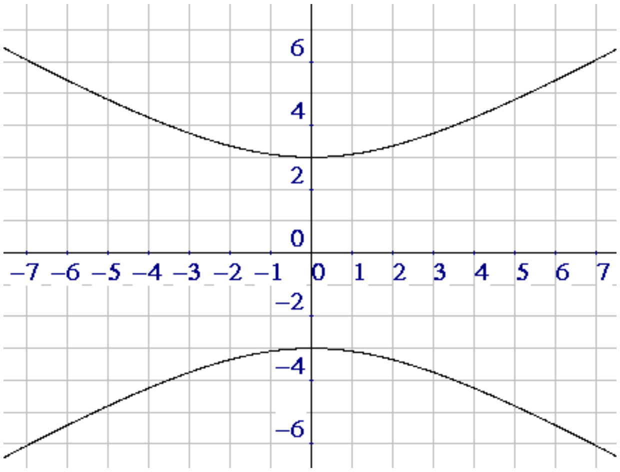
Not a function

No x-intercepts; y-intercepts: (0, 4) and (0, −4)

Symmetry: all three

Asymptotes: $y = 3x$ and $y = -3x$

9.



Domain = $\mathbb{R}$

Range = $(-\infty\ -3] \cup [3, \infty)$

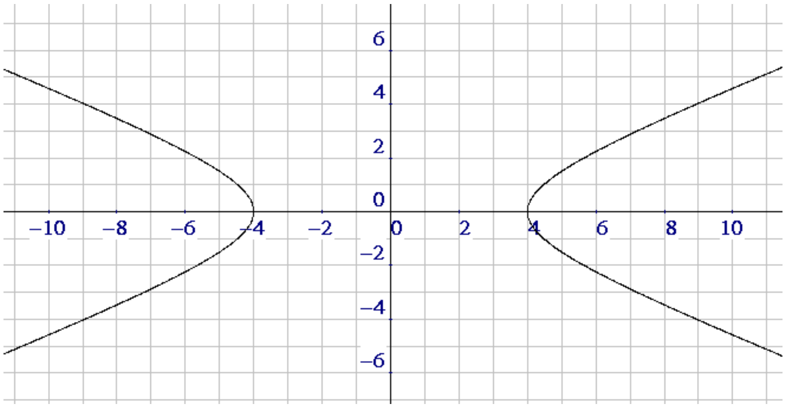
Not a function

No x-intercepts

y-intercepts: (0, 3) and (0, −3)

Symmetry: all three

Asymptotes: $y = \pm\frac{3}{4}x$

10. 

Domain = $(-\infty, -4] \cup [4, \infty)$ Range = $\mathbb{R}$ Not a function

Intercepts: $(-4, 0)$ and $(4, 0)$ Symmetry: all three

Slant asymptotes: $y = \pm\frac{1}{2}x$

11. a. parabola b. line c. ellipse d. hyperbola

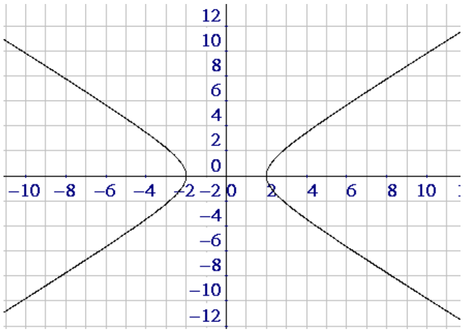
e. (vertical) line f. (horizontal) line g. circle

12. a. none b. hyperbola c. ellipse
d. parabola e. hyperbola f. hyperbola

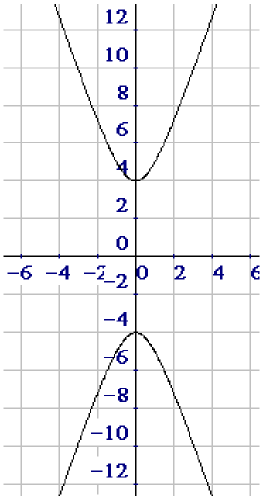
13. a. $\left(\pm\sqrt{\frac{2}{3}},\ 0\right)$ b. $\left(0,\ \pm\sqrt{\frac{3}{10}}\right)$ c. $(-\infty, -12] \cup [12, \infty)$ d. $\mathbb{R}$

e. $y = \pm 7x$ f. $y = \pm\frac{1}{3}x$ g. x-axis, y-axis, and origin

14. $(\pm 2, 0)$ Dom = $(-\infty, -2] \cup [2, \infty)$ Rg = $\mathbb{R}$ asym: $y = \pm x$
all 3 symmetries



15. $y = \pm 3x$; $(-\infty, -4] \cup [4, \infty)$



16. a. F b. F c. T d. T e. T f. T g. F

"Effort only fully
releases its reward
after a person refuses to quit."

Napoleon Hill

CH NN – INEQUALITIES

You must score between 80% and 89% to get a B in your math class.

You must be at least 18 years of age to vote.

You can be no taller than 48 inches to play in the park.

These are all examples of quantities being greater than something or less than something (or between two somethings). Since they are not equalities, they are called **inequalities**.

☐ *NOTATION*

We know that 5 is bigger than 3, which we can write as "**5 > 3**." The symbol ">" can also be read as "is larger than" or "is greater than."

$>$ means "is greater than"

$<$ means "is less than"

$\geq$ means "is greater than or equal to"

$\leq$ means "is less than or equal to"

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is smaller than 5. This is written "**3 < 5**."

The symbol "≥" can read "is greater than or equal to." For example, **9 ≥ 7** because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than or equal to 7.)

Believe it or not, **$12 \geq 12$** is also a true statement — after all, since 12 equals 12, it's certainly the case that 12 is greater than <u>or</u> equal to 12.

The symbol "≤" is read "is less than or equal to." A couple of examples are $6 \leq 10$ and $8 \leq 8$.

Homework

1. True/False:

   a. $7 > 3$ b. $-2 < 1$ c. $13 \geq 13$ d. $-9 \leq -9$

   e. $12 \geq 9$ f. $-18 \leq -20$ g. $\pi > 0$ h. $-\sqrt{2} \leq 0$

2. Express each statement as an inequality:

   a. Your age, a, must be at least 18 years.

   b. Your height, h, can be no taller than 48 inches.

   c. Your years of experience, y, must exceed 10 years.

   d. The number of driving tickets, t, must be fewer than 5.

   e. The mean, μ (Greek letter mu), must be at least 75.

   f. The standard deviation, σ (Greek letter sigma), must be no more than 10.

   g. The energy, E, must be more than 100.

   h. The mass, m, must be less than 3.7.

❑ *THE BASIC PRINCIPLE OF SOLVING INEQUALITIES*

To solve an inequality such as $-2x + 7 \geq -4$, we have to perform the operations necessary to isolate the x. We will accomplish this goal by using the standard *do the same thing to each side* rule, but since we're talking about an inequality — not an equation — we have to be very careful. The following experiment should illustrate the potential problems, and give us the techniques to overcome these problems.

Let's perform six experiments on a given inequality. Consider the true statement

$$4 < 6$$

i. Add 10 to each side: $4 + \mathbf{10} < 6 + \mathbf{10}$

$\Rightarrow 14 < 16$ ✓

ii. Subtract 3 from each side: $4 - \mathbf{3} < 6 - \mathbf{3}$

$\Rightarrow 1 < 3$ ✓

iii. Multiply each side by 5: $4(\mathbf{5}) < 6(\mathbf{5})$

$\Rightarrow 20 < 30$ ✓

iv. Divide each side by 2: $\frac{4}{2} < \frac{6}{2}$

$\Rightarrow 2 < 3$ ✓

v. Multiply each side by –3: $4(-3) < 6(-3)$

$\Rightarrow -12 < -18$ ✗

vi. Divide each side by –2: $\frac{4}{-2} < \frac{6}{-2}$

$\Rightarrow -2 < -3$ ✗

What can we deduce (conclude) from these six calculations? The first two show that adding the same number to each side of an inequality, or subtracting the same number from each side of an inequality, are both things we can do without any issues; they lead to a true inequality.

Calculations iii. and iv. indicate that multiplying or dividing each side of an inequality by a positive number leads to an inequality which is just as valid as the original one.

Unfortunately, the last two cases — multiplying or dividing by a negative number — have led to false statements. So in these two scenarios, we must reverse the order of the inequality sign (flip it around) in order to maintain a true statement. Problem solved!

These numerical experiments, by no means a complete proof of the principle we are about to state, are convincing enough for me.

The Basic Principle of Solving Inequalities

Only if you multiply or divide each side of an inequality by a *negative* number, must you reverse the inequality symbol.

EXAMPLES: **Solve each inequality:**

A. Solve the inequality: $x + 3 > 4$

Subtract 3 from each side $x > 1$

Note: The inequality symbol was not reversed.

B. Solve the inequality: $-2n - 9 \leq 13$

Add 9 to each side: $-2n \leq 22$
(The inequality symbol was not reversed.)

Divide each side by –2: $n \geq -11$

This time the inequality symbol <u>was reversed</u>, since we divided each side of the inequality by a negative number.

C. Solve the inequality: $\boldsymbol{\frac{u}{5}+3 < -4}$

Subtract 3 from each side: $\frac{u}{5} < -7$

Multiply each side by 5: $\boldsymbol{u < -35}$

Neither operation required reversing the inequality.

Homework

3. Solve each inequality:

   a. $x + 7 > -10$ b. $x - 3 \le 5$ c. $2x \ge 14$

   d. $-3x < -42$ e. $\frac{x}{8} > -3$ f. $\frac{x}{-5} \ge 10$

4. Solve for y: $6(y - 5) - (y + 1) \le 12y + 21$

5. Solve for u: $-2(7 + 3u) - (1 - u) > 2u - 10$

6. a. Solve for x: $\frac{-3x+5}{-2} + 9 > 16$ Note: The process is more important than the answer. Do you know what this means?

   b. Solve for a: $\frac{7a-5}{-3} - 12 < 12$

   c. Solve for y: $\frac{9-y}{-2} \ge -2$

   d. Solve for n: $\frac{-14-2n}{12} \le -1$

☐ *UNDERSTANDING DOUBLE INEQUALITIES*

Suppose Edwin wants to take part in a political poll. The pollster only cares about voters whose age, A, is between 25 and 40 (including the 25 and the 40). Here's one way to write this requirement using algebra:

$A \geq 25$ **and** $A \leq 40$ (a double inequality using **and**)

By the way, the word **and** between the two inequalities is crucial; Edwin must be at least 25 **and** at most 40. Do you see that replacing the **and** with **or** would change the meaning entirely?

Here's the other way to write that double inequality — we sandwich the variable A between the lower limit (25) and the upper limit (40):

$$\mathbf{25 \leq A \leq 40}$$

This can be read in two equivalent ways:

1. $A \geq 25$ **and** $A \leq 40$ [Note that $25 \leq A$ is the same as $A \geq 25$.]
2. A is ***between*** 25 and 40, including the 25 and including the 40.

For a second example, the phrase "*x must be between 0 and 10, but not equal to either 0 or 10,*" is written

$$\mathbf{0 < x < 10}$$

Homework

7. True/False:
   a. 7 satisfies the double inequality: $0 < x < 9$.
   b. 0 satisfies the double inequality: $0 < x < 9$.
   c. 10 satisfies the double inequality: $-5 \leq x \leq 10$.
   d. 0 satisfies the double inequality: $0 \leq x < \pi$.

e. 2π satisfies the double inequality: $0 < x < 7$.

f. 100 satisfies the double inequality: $100 \le x < 101$.

8. a. Name the only solution of the double inequality

$$23 \le x \le 23$$

b. Find all values of x which satisfy the double inequality

$$50 < x < 50$$

9. Express each situation as a *double inequality*:

a. The weight, w, must be between 20 lb and 50 lb, including the 20 and the 50.

b. The score, s, must be between 100 pts and 300 pts, excluding the 100 and the 300.

c. The commission, C, must be between \$2000 and \$5750, inclusive of the endpoints.

d. The distance, d, must be between 123 m and 250 m, exclusive of the endpoints.

☐ *TO ∞ AND BEYOND*

A. Determine the values of *a* which satisfy the compound inequality:

$$a \ge 25 \;\; \textbf{or} \;\; a \le 40$$

B. Marty was trying to solve the inequality

$$ax + b > c$$

for *x*, and wrote the following:

$$ax + b > c$$

$$\Rightarrow \quad ax > c - b$$

$$\Rightarrow \quad \boxed{x > \frac{c - b}{a}}$$

Explain the fallacy (faulty logic) in Marty's reasoning.

Solutions

1. a. T b. T c. T d. T e. T f. F
g. T h. T

2. a. $a \geq 18$ b. $h \leq 48$ c. $y > 10$ d. $t < 5$
e. $\mu \geq 75$ f. $\sigma \leq 10$ g. $E > 100$ h. $m < 3.7$

3. a. $x > -17$ b. $x \leq 8$ c. $x \geq 7$
d. $x > 14$ e. $x > -24$ f. $x \leq -50$

4. $y \geq -\frac{52}{7}$

5. $u < -\frac{5}{7}$

6. a. $x > \frac{19}{3}$ b. $a < -\frac{67}{7}$
c. $y \geq 5$ d. $n \geq -1$

7. a. True b. False c. True
d. True e. True f. True

8. I'd like to know what you think.

9. a. $20 \leq w \leq 50$ b. $100 < s < 300$
c. $2000 \leq C \leq 5750$ d. $123 < d < 250$

> "Do not judge me by my successes. Judge me by how many times I fell down and got back up again."
>
> – *Nelson Mandela*

CH XX – INTERVAL NOTATION

Consider all the real numbers greater than 3. One simple way to express this set of numbers is the following ***inequality***:

$x > 3$ [Do you see that this inequality can also be written "3 < x"?]

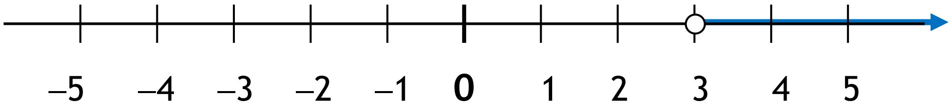
But there are other ways to represent the inequality $x > 3$, ways that might be easier to visualize.

☐ *EXAMPLES*

First Example: Let's continue with the inequality stated in the introduction:

$x > 3$

A great way to visualize this inequality is to ***graph*** this set of numbers on a number line:



Notice that we put an "open dot" at $x = 3$ to show that the 3 is not part of the set of numbers. But the arrow goes infinitely to the right because $x > 3$ is the set of numbers greater than 3.

Note that the numbers 3.01, π, 17, and 200 are part of the set; but the numbers –5, 0, 2.5, and 3 are not part of the set.

And a third way to denote this interval is called ***interval notation***, and for this inequality, we write:

$(3, \infty)$

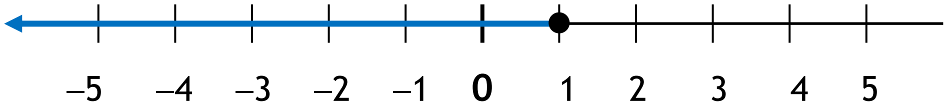
The parenthesis next to the 3 is analogous to the "open dot" on the graph — it means <u>exclude</u> the endpoint. And you <u>always</u> use a parenthesis at the ∞ or –∞ end of an interval because ∞ is not really a number, so you can't possibly include it.

<u>Second Example:</u> Now we consider all the real numbers less than or equal to 1. This set of numbers can be written as an inequality like this:

$$x \le 1$$

Note that this inequality can also be expressed as: $1 \ge x$

As a graph on a number line, we write:



The "solid dot" is used to show that $x = 1$ <u>is</u> part of the set of numbers. And since x must be less than or equal to 1, the arrow goes infinitely to the left. Either as an inequality or a graph, you should see that the numbers –3, –1.1, $\frac{7}{8}$, and 1 <u>are</u> part of the set, while the numbers 1.001 and $\sqrt{2}$ are <u>not</u> part of the set.

As for interval notation, we write

$$(-\infty, 1]$$

The (square) bracket next to the 1 is analogous to the "solid dot" on the graph — it means <u>include</u> the endpoint. And, as mentioned before, always use a parenthesis with ± ∞, since you can't ever "get" to ∞.

<u>Third Example:</u> Now it's time for an interval that represents all the numbers <u>between</u> two numbers. Consider the double inequality

$$-2 \le x < 5$$

The inequality can also be read as "all real numbers that are both greater than or equal to 2 AND also less than 5.

This can be read as “all real numbers between –2 and 5, including the –2, but excluding the 5.”

As for interval notation and a graph, here they are:

$[-2, 5)$

Note that some numbers in the interval are –2, 0, $\sqrt{24}$, and 4.9999. But the numbers –2.1, 5, and 2π are not in the interval.

Fourth Example: Sometimes answers to an inequality problem end up looking something like this:

$$x < 3 \text{ OR } x \geq 5$$

This means pretty much what it says: x can be less than 3, or it can be greater than or equal to 5. As long as x satisfies (at least) one of the two conditions, it's part of the answer. So some x's that satisfy the inequality are 0, 2.9, 5, and 3π. On the other hand, some numbers that do NOT satisfy it are 3, π, and 4.99.

How do we express this in interval notation? Like this, using the **union** symbol:

$$(-\infty, 3) \cup [5, \infty)$$

The *union* symbol, ∪, just means putting all the elements of the two sets into one big set; for example,

{a, b} ∪ {a, c, d} = {a, b, c, d}

Note that the “overlap” of ‘a’ is listed only once in the final set.

And as a graph, we write:

Can you see that the interval $(-\infty, \infty)$ is the same as the entire set of real numbers?

$$(-\infty, \infty) = \mathbb{R}$$

In fact, using the *union* notation, we can also write

$$\mathbb{R} = \text{Rationals} \cup \text{Irrationals}$$

where the Rational numbers are all decimals that do repeat, while the Irrational numbers are the decimals that do not repeat. So $\mathbb{R}$ is simply the set of ALL decimals.

❑ SUMMARY OF INEQUALITIES AND INTERVALS

Inequality	Interval
$x > b$	(b, ∞)
$x \ge b$	$[b, \infty)$
$x < a$	$(-\infty, a)$
$x \le a$	$(-\infty, a]$
$a \le x \le b$ which means $x \ge a$ AND $x \le b$, that is, x is *between* a and b, including both the a and the b.	$[a, b]$
$x < a$ OR $x > b$	$(-\infty, a) \cup (b, \infty)$

Homework

1. Convert each inequality to *interval notation*:

a. $x > 2$ b. $x \le 5$ c. $-1 \le x < 6$ d. $x < -3$ OR $x > 0$

2. Convert each interval to an *inequality*:

a. $[3, \infty)$ b. $(-\infty, -5)$ c. $(-1, 8]$ d. $(-\infty, -2] \cup (7, \infty)$

3. Prove that the statement

$$x < 3 \text{ AND } x \ge 5$$

consists of absolutely nothing, the null set, $\varnothing$.

4. What is the relationship between the notation $\mathbb{R}$ and the notation $\boldsymbol{(-\infty, \infty)}$?

Solutions

1. a. $(2, \infty)$ b. $(-\infty, 5]$ c. $[-1, 6)$ d. $(-\infty, -3) \cup (0, \infty)$

2. a. $x \ge 3$ b. $x < -5$ c. $-1 < x \le 8$ d. $x \le -2$ OR $x > 7$

3. There is <u>no</u> number that is both less than 3 <u>and</u> greater than or equal to 5. The two sets of numbers are disjoint, and so their intersection is $\varnothing$.

4. They both mean exactly the same thing: the set of all **real numbers**.

“Develop a passion for learning.
If you do, you will never
cease to grow.”

Anthony J. D’Angelo

CH NN – LIMITS

The title of this book is *From Logic to Limits*, and the first chapter is entitled **Logic**. So, obviously, this last chapter must be called **Limits**.

EXAMPLE 1:

Infinity may not be a number, but it's a pretty interesting concept. Although a variable x cannot actually be infinity, we can nevertheless ask ourselves: What happens to a function as x *approaches* infinity as a limit?

∞

Consider the function $y = \frac{1}{x}$.

If $x = 10$, then $y = \frac{1}{10}$.

If $x = 500$, then $y = \frac{1}{500}$.

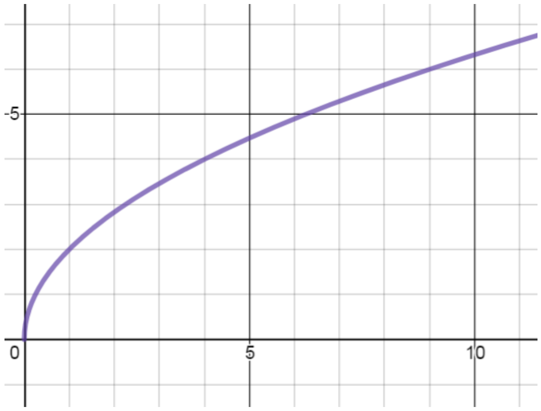
And if we let x be really big, say $x = 1{,}000{,}000$, then $y = \frac{1}{1{,}000{,}000}$.

It seems that as x grows larger and larger, the y-values are getting closer and closer to zero. In fact, we can say that x is approaching infinity, and this is causing y to approach zero. In other words,

As x approaches infinity, y approaches zero:

$$\text{As } x \to \infty,\ y \to 0$$

EXAMPLE 2: **Analyze the following limits for the graph:**



You might recognize this as being the graph of the function $y = \sqrt{x}$.

As $x \to \infty$, $y \to \infty$: Imagine x growing larger and larger. For example, if $x = 100$, then $y = 10$; if $x = 10{,}000$, then $y = 100$. It's clear that, as x grows larger and larger, so does y (though not nearly at the same speed as the x). On the graph, as we move farther and farther to the right, the graph keeps getting higher and higher. In fact, we can rise as high as we desire by choosing x appropriately large. This is why we say: As $x \to \infty$, $y \to \boldsymbol{\infty}$.

As $x \to 0$, $y \to 0$: Note that, since $\sqrt{x}$ is defined only for $x \geq 0$, we'll make sure we use positive values of x that are getting smaller and smaller (approaching 0).

The table indicates that $\sqrt{x}$ approaches **0** also.

x	$\sqrt{x}$
1	1
0.5	0.707
0.25	0.5
0.01	0.1
0.001	0.032
0.0002	0.004

$\downarrow$ $\downarrow$

0 **0**

<u>Note:</u> It makes no difference at all what happens when *x* = 0. In a limit, we care only about what's happening as *x* gets closer and closer to the number it's approaching. In fact, we don't care whether or not *x* is even allowed to be 0.

As $x \to 9$, $y \to 3$: As *x* gets closer and closer to 9, *y* gets closer and closer to 3:

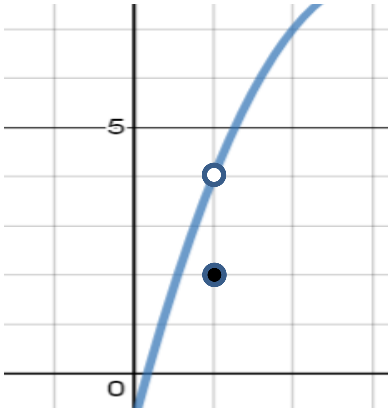
x	8	8.3	8.5	8.8	8.9	8.99	$\longrightarrow$ 9
$\sqrt{x}$	2.828	2.88	2.915	2.966	2.983	2.998	$\longrightarrow$ 3

Final Note This last limit shows that as *x* approaches 9, *y* approaches 3. It is also the case that when *x* = 9, *y* = 3. Yet, this fact has NOTHING to do with the limit. Even if *y* were a hundred trillion when *x* = 3, the limit statement would still be correct.

> "The educated differ from
> the uneducated
> as much as the living from
> the dead."
>
> Αριστοτλε (Aristotle)

Bottom line: Functional values are one thing, **a**nd limits are another. The following example might clarify this critical distinction. It also gives us limits which do <u>not</u> involve infinity.

EXAMPLE 3: **Let f be defined by the graph**



a. $f(1) =$ _____

b. As $x \to 1, y \to$ _____

Solution:

a. Looking at the solid dot part of the graph, we conclude that $f(1) = \mathbf{2}$.

b. As x approaches 1, the graph is heading toward a y-value of 4. The open circle at the point (1, 4) means that x never reaches 1 on the curve. Still, as x approaches 1, y approaches a value of **4**. That is, as $\boldsymbol{x \to 1, y \to 4}$

To summarize, at $x = 1$, the function value is $\underline{2}$. But as $x \to 1$, $y \to \underline{4}$. So we note again that calculating limits has nothing at all to do with what might happen if x actually becomes the value it's approaching. In fact, in the graph of this example, if we were to remove the solid dot, the answer to part a. would change to "Undefined," but the limit would remain exactly as it is.

CH NN – LINEAR MODELING USING GIVEN INFORMATION

❑ INTRODUCTION

Sometimes it's not too hard to create a formula from words. For example the phrase *density is the quotient of mass and volume* translates to the formula $d = \frac{m}{V}$. But this method of formula creation is useful only if we comprehend the words that need to be translated into Algebra.

More often, we're given a *situation* that needs translation. For example, if we buy 12 lbs of cashews at a unit price of $14/lb, and we wish to determine the total cost, we have to understand that multiplication does the trick; no one's going to tell you that — you have to reason it out yourself. So the total cost is

12 lbs × $14/lb = <u>$168</u>

That's fine for one or a few calculations. But if you need to program a calculator, or perhaps a spreadsheet, or maybe a programming language, you need to teach the machine what to do, and that can only be done with *variables*. So in the cashew problem above, we could let Q stand for the quantity (# of pounds) of cashews, let P be the unit price (the price of 1 pound) of the cashews, and let C be the total cost. Again, if we understand that the key to the formula will be multiplication, we can write a formula that can be entered into a computer (using *, the computer symbol for multiplication):

$$C = P * Q$$ [Cost = Unit Price times Quantity]

❑ JOINING THE CLUB

EXAMPLE : **To join the Model Railroad Club, a member must pay an up-front fee of $25 to join, and then pay $10 per month for each month in the club.**

a. Find the total cost for someone to be a club member for 8 months.

b. Find a formula for someone to be a club member for *m* months.

c. Use your formula to calculate the total to be in the club for 36 months.

Solution: Let's begin with a simple table that shows the cost of membership for various months:

Months	0	1	2	3	4	5	6	7	8
Cost	$25	$35	$45	$55	$65	$75	$85	$95	$105

a. The cost, C, is $25 plus 8 months at $10 per month:

$$C = 25 + 10(8) = 25 + 80 = 105$$

Therefore, the cost of 8 months is **$105**.

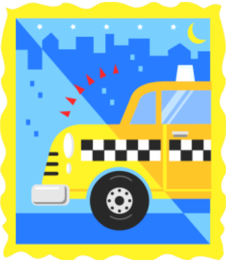
b. Just change the 8 in part a. to the variable m:

$$C = 25 + 10m, \text{ or } \mathbf{C = 10m + 25}$$

c. Using the formula in part b., we get a cost of

$$C = 10m + 25 = 10(\mathbf{36}) + 25 = 360 + 25 = \mathbf{\$385}$$

Homework



1. a. It costs \$2 per mile to take a taxi. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Use your formula to calculate the total cost of a 20-mile trip.

2. Sprint offers a monthly phone plan that has a base fee of \$20, plus \$0.25 for each minute of use. Write a formula for the monthly cost, C, if you used the phone for m minutes in a month.

3. a. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. Calculate the total cost of a 21-mile taxi ride.

   b. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   c. Use your new formula from part b. to calculate the total cost of a 15-mile trip.

4. It costs \$7 for the first mile of a taxi ride, and \$4 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

5. The ultimate question on the taxi ride:

   Let C = the total cost of the taxi ride

   f = the cost of the first mile

   a = the cost of each mile after the first

   m = the total miles traveled

   Write a formula for the total cost of a taxi ride.

6. a. QRS, Incorporated sold 5000 widgets last month at a unit selling price of \$45/widget. Calculate the revenue obtained. Hint: Revenue is the total amount of money obtained.

   b. STU, Inc. sold w widgets last month at a unit selling price of \$$p$/widget. Create a formula for the revenue, R, obtained.

7. a. WXY, Inc. produced 600 widgets last quarter, and it costs \$50 to produce each widget. (\$50/widget is called the *unit cost*). In addition, fixed costs (rent, utilities, salaries, etc.) totaled \$2,400 last quarter. Calculate the total cost of producing the 600 widgets last quarter.

   b. Let w = the number of widgets produced

      c = unit cost to produce one widget

      f = fixed costs (rent, utilities, salaries, etc.)

      E = total expense to produce the w widgets

   Create a formula for the total expense, E, of producing w widgets.

8. Ernie earns a base salary of \$2000/month and a commission of \$50 for each iPad he sells. Find Ernie's total salary during a month in which he sold 75 iPads. Construct a formula which would give Ernie's total salary, S, during a month if he sold i iPads.

9. Andrew wants to place a classified ad in the Chronicle to sell his old iPad .

   a. The first 12 words cost \$7.50, and each additional word costs \$0.30. Find the total cost of an ad containing 20 words.

   b. The first 15 words cost \$9.75, and each additional word costs \$0.40. Construct a formula to calculate C, the total cost of an ad containing w words. [Assume that w is at least 15.]

10. A tole painter buys a piece of wood for $\$w$, paints it, and then sets the selling price at 4 times the cost of the wood. If the painter gets 75% of the selling price (the rest goes to the salesperson), find the tole painter's NET profit (money earned less expenses).

Review Problems

11. a. Suppose that the Foothill Middle School library currently has 7000 books, and is buying 150 more books per year. How many books will the library have 4 years from now?

    b. Write a formula that will give the number of books, B, that the library will have y years from now.

12. You must pay $75 up front to join the Painting Club, and then pay $35 per month for each month that you're a member of the club. Create a formula that will give the total cost, C, to be in the club for m months.

13. It costs $7.50 for the first mile of a taxi ride, and $3.25 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

14. The Speedy Car Rental Co. charges $35/day and $0.15/mile to rent a car. Let d represent the number of days you rent the car, and let m represent the number of miles driven. Find a formula for C, the total rental cost.

❑ *TO ∞ AND BEYOND*

Placing a Classified Ad

The first n words cost $\$I$, while each additional word costs $\$A$. Find the total cost, C, of an ad containing w words. You may assume that $w \geq n$.

Solutions

1. a. $C = 2m$ b. $C = 2m = 2(20) = \mathbf{\$40}$

2. $C = 20 + 0.25m$ [or, $C = 0.25m + 20$]

3. a. $3 + 2(20) = 3 + 40 = \mathbf{\$43}$
b. $C = 3 + 2(m - 1)$, or $\boldsymbol{C = 2(m - 1) + 3}$
c. $C = 2(m - 1) + 3 = 2(15 - 1) + 3 = 2(14) + 3 = 28 + 3 = \mathbf{\$31}$

4. $C = 7 + 4(m - 1)$ or $C = 4(m - 1) + 7$

5. $C = f + a(m - 1)$ or $C = a(m - 1) + f$

6. a. $225,000 b. $R = pw$

7. a. $600(50) + 2{,}400 = \mathbf{\$32{,}400}$ b. $E = cw + f$

8. $\$2000 + \$50(75) = \mathbf{\$5{,}750}$
In general, $S = 2000 + 50(i)$, or $\boldsymbol{S = 50i + 2000}$

9. a. $9.90 b. $C = 9.75 + 0.40(w - 15)$

10. $0.75(4w) - w$

11. a. $7000 + 4(150) =$ <u>7600 books</u>
b. $B = 150y + 7000$ or $B = 7000 + 150y$

12. The cost would be \$75 plus \$35 for each month in the club.
That's \$75 + \$35 × the number of months in the club.
That's $C = 75 + 35m$, which could be written $\underline{C = 35m + 75}$

13. $C = 7.50 + 3.25(m - 1)$

14. $C = 35d + 0.15m$

"Never doubt that a
small group of thoughtful,
committed citizens can
change the world;
indeed it is the only thing
that ever has."

Margaret Mead, Anthropologist

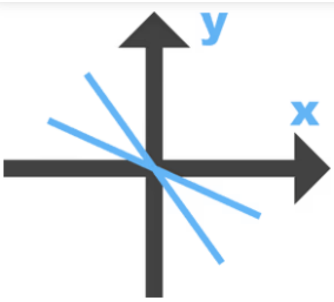
CH NN – LINEAR MODELING USING THE EQUATION OF A LINE

For the problems in this chapter you may use either of the two forms of a line we've studied: the slope-intercept form,

$$y = mx + b$$

or the point-slope form,

$$y - y_1 = m(x - x_1).$$



I believe you'll find the point-slope form easier because problems in this chapter will provide us with a pair of points on the line, not the *y*-intercept. In addition, this form is the preferred method in calculus.

EXAMPLE 1: **On day 1 the amount, *A*, of digoxin in the body was 3.2 milligrams (mg). On day 7 the amount was 17 mg. Assume that the amount of digoxin is a linear function of the day, *d*.**

a. Find the linear function that expresses the amount, *A*, as a function of the day, *d*.

b. What does the formula predict for the amount of digoxin on day 10?

c. On what day will *A* be 51.5 mg?

d. What does the *slope* of the line represent?

e. Solve the formula for *d*.

Solution: Don't be overwhelmed — almost all of the work is in part **a.** We start by organizing the given data into ordered pairs of the form (d, A), where d is the day and A is the amount of digoxin:

$$(1, 3.2) \quad (7, 17)$$

Note: d is basically the x, and A is basically the y.

a. The best method we have for finding the equation of the line through the two points is the point-slope method, whose formula is

$$A - A_1 = m(d - d_1)$$

where m is the slope and (d_1, A_1) is any point on the line.

So the only calculation we need is the slope:

$$m = \frac{\Delta A}{\Delta d} \qquad \left[\text{just like } \frac{\Delta y}{\Delta x}\right]$$

$$= \frac{17 - 3.2}{7 - 1}$$

$$= \frac{13.8}{6}$$

$$= \mathbf{2.3}$$

We can now choose either point; let's choose (7, 17):

$$A - 17 = \mathbf{2.3}(d - 7)$$

This a fine way to write the linear function, but it's actually better to rewrite it in slope-intercept form — this means solve for A and simplify:

$$A - 17 = 2.3(d - 7) \qquad \text{(the point-slope form)}$$

$$\Rightarrow \qquad A - 17 = 2.3d - 16.1 \qquad \text{(distribute)}$$

$$\Rightarrow \quad A = 2.3d - 16.1 + 17 \qquad \text{(add 17 to each side)}$$

$$\Rightarrow \quad \boxed{A = 2.3d + 0.9} \qquad \text{(the slope-intercept form)}$$

b. Just let $d = 10$ in the formula:

$$A = 2.3d + 0.9 = 2.3(\mathbf{10}) + 0.9 = 23 + 0.9 = \mathbf{23.09\ mg}$$

c. Now let $A = 51.5$ in the formula:

$$\mathbf{51.5} = 2.3d + 0.9 \Rightarrow 50.6 = 2.3d \Rightarrow d = 22 \Rightarrow \textbf{Day 22}$$

d. The slope, 2.3, represents the average daily increase in the amount of digoxin.

e.

$$A = 2.3d + 0.9$$

$$\Rightarrow \quad A - 0.9 = 2.3d$$

$$\Rightarrow \quad \boldsymbol{d = \frac{A - 0.9}{2.3}}$$

Homework

1. On day 5 the height of a plant is measured to be 38.1 cm. On day 9 the height has increased to 63.7 cm.

   a. Find the linear formula that expresses the height *h* of a plant as a function of the day, *d*.

   b. Use your formula to predict the height of the tree on day 11.

   c. On what day is the plant 127.7 cm tall?

   d. What does the slope of your line represent?

   e. Solve the formula for *d*.

2. During a recent snowstorm, there were 12 inches of snow on the ground at 7 p.m., and there were 19 inches at 11 p.m. If you were to graph this data (with time on the horizontal axis and snow on the vertical axis), calculate the slope, and then describe what the slope of the line represents in the context of this problem.

3. A scientist measured the pressure at a depth of 2 ft and determined that it was 15.59 lbs/in^2 (pounds per square inch). At a depth of 10 ft the pressure was 19.15 lbs/in^2.

   a. Find the linear formula that expresses the pressure P as a function of the depth d. [Round to 3 digits.]

   b. Use your formula to predict the pressure at a depth of 100 ft.

   c. At what depth would we find a pressure of 85.9 lbs/in^2?

   d. What does the 0.445 in your formula represent?

   e. According to the formula, what would be the pressure at sea level ($d = 0$)? Does this make any sense?

   f. Solve the formula for d.

Solutions

1. a. $h = 6.4d + 6.1$ b. 76.5 cm c. day 19

d. It's the amount the plant grows each day (on average)

e. $d = \frac{h-1}{6.4}$

2. $m = \frac{7}{4}$ in/hr

The slope represents the average hourly rate of change in the snow level. That is, it's a measure of how fast it's snowing.

3. a. $P = 0.445d + 14.7$

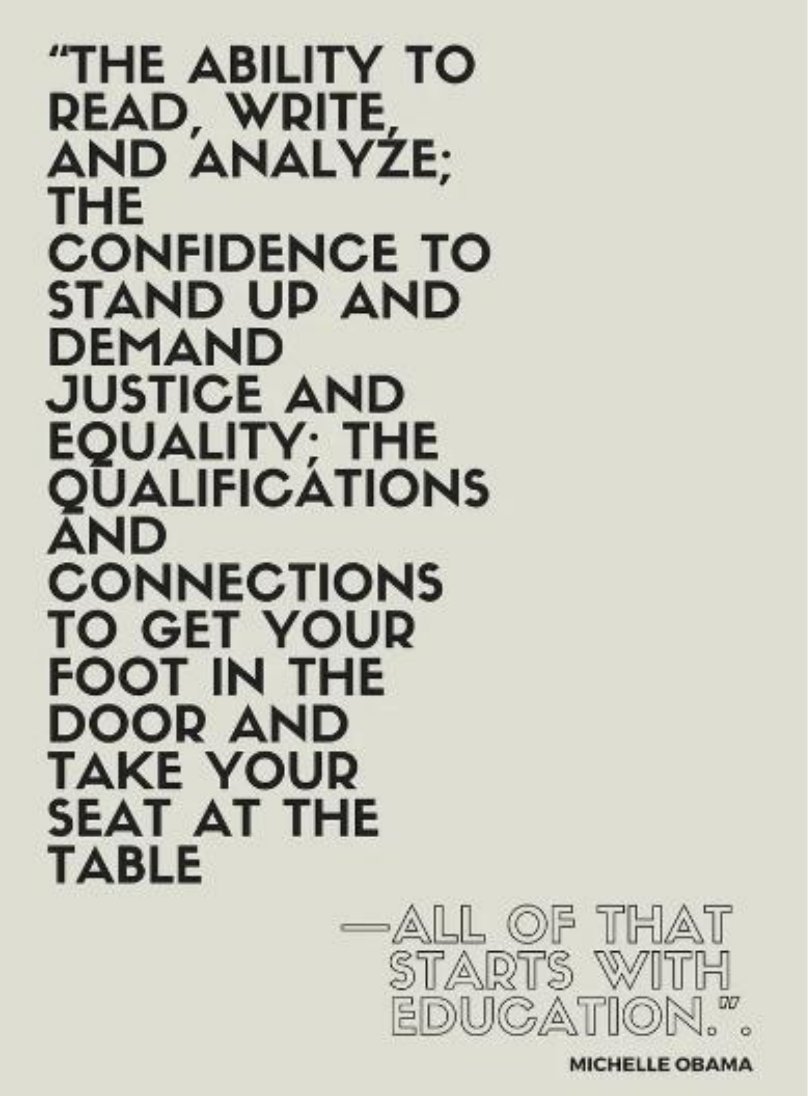
b. 59.2 lbs/in^2

c. 160 ft

d. The increase in pressure for each additional foot below sea level.

e. The pressure would be 14.7 lbs/in^2. It's not an error. Can you account for the pressure? Hint: It's not water pressure.

f. $d = \frac{P - 14.7}{0.445}$



“THE ABILITY TO READ, WRITE, AND ANALYZE; THE CONFIDENCE TO STAND UP AND DEMAND JUSTICE AND EQUALITY; THE QUALIFICATIONS AND CONNECTIONS TO GET YOUR FOOT IN THE DOOR AND TAKE YOUR SEAT AT THE TABLE

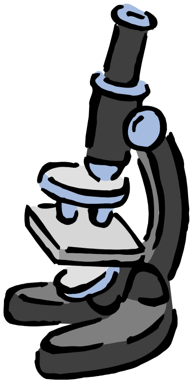
—ALL OF THAT STARTS WITH EDUCATION.”.

MICHELLE OBAMA

CH XX – FINDING THE EQUATION OF A LINE, $y = mx + b$

❑ *INTRODUCTION*

Given an equation like $y = 7x + 3$, we've learned that its graph is a line with a slope of 7 and a y-intercept of (0, 3), and we certainly could find other points on the line. In this chapter we turn the tables: Given some information about the line, we try to seek the equation of the line. For you CSI fans, it's like using blood and hair samples to determine the identity of the killer.



❑ *FINDING THE LINE GIVEN SLOPE AND Y-INTERCEPT*

EXAMPLE 1: **Find the equation of the line which has a slope of –3 and whose y-intercept is the point (0, 9).**

Solution: This is the easiest possible problem asking us to find the equation of a line. Here's why: The slope-intercept form of a line we've learned is

$$y = mx + b$$

where m is the slope and $(0, b)$ is the y-intercept. To "fill in the blanks" of this equation, we need m and b. Now look at what's given to us in the problem: the slope and the y-intercept. That is, we're told that $m = -3$ and $b = 9$. We're done!

$$\boxed{y = -3x + 9}$$

☐ *FINDING THE LINE GIVEN THE SLOPE AND A POINT*

EXAMPLE 2: **Find the equation of the line which has a slope of 7 and which passes through the point (–5, 3).**

Solution: The line equation we are using is $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept. In this example, we are given the slope of 7. That's good.

And so the line equation $y = mx + b$

becomes $y = 7x + b$ (putting the 7 in for slope)

But did the problem give us the y-intercept? No; instead, we are given a point on the line, but I know that it's not the y-intercept. (How can I be so sure?) So now our goal is to find the value of b.

Consider that the problem tells us that the point (–5, 3) is on the line. Therefore, the point must work in the equation we just wrote, $y = 7x + b$. So we plug –5 in for x and plug 3 in for y:

$$y = 7x + b \quad \text{(our line with the slope plugged in)}$$

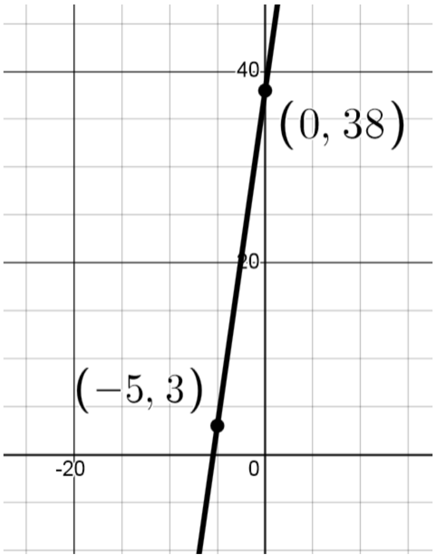
$$\Rightarrow \mathbf{3} = 7(\mathbf{-5}) + b \quad \text{(since (–5, 3) lies on the line)}$$

$$\Rightarrow 3 = -35 + b \quad \text{(multiply)}$$

$$\Rightarrow b = 38 \quad \text{(solve for } b\text{)}$$

Now we put it all together. The value of m is 7 (given in the problem) and the value of b is 38 (we just calculated it). We get our final answer:

$$\boxed{y = 7x + 38}$$



Homework

1. Find the equation of the line with the given slope and passing through the given point:

a. $m = -3$; $(8, -16)$

b. $m = 4$; $(-1, -11)$

c. $m = 2$; $(0, -10)$

d. $m = -3$; $(3, -2)$

e. $m = 9$; $(2, 18)$

f. $m = -1$; $(-3, 11)$

g. $m = 1$; $(5, 5)$

h. $m = 7$; $(0, 10)$

i. $m = -8$; $(-2, 29)$

j. $m = 1$; $(10, -89)$

❑ *FINDING THE LINE GIVEN TWO POINTS*

EXAMPLE 3: **Find the equation of the line passing through the points (–1, 3) and (8, –15).**

Solution: This problem will really test our deductive skills. Let's begin with the slope-intercept form of a line:

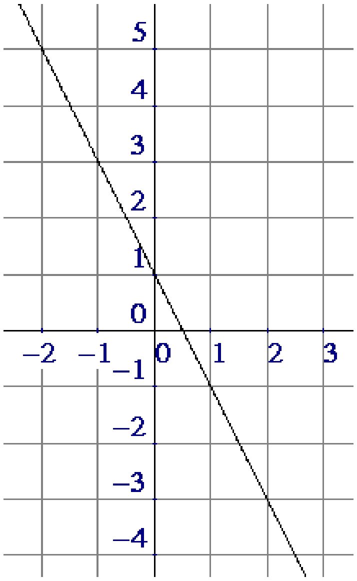
$$y = mx + b$$ [we need the slope and the *y*-intercept]

Did the problem tell us what the slope is? No. Did the problem give us the *y*-intercept? No. This is not good — how can we possibly solve this problem? Well, even though the slope was not handed to us on a silver platter, we can use the two given points on the line to calculate the slope, using our $m = \frac{\Delta y}{\Delta x}$ formula. So, using the given points (–1, 3) and (8, –15), we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{3-(-15)}{-1-8} = \frac{3+15}{-1-8} = \frac{18}{-9} = -2$$

We can now write our line as

$$y = -2x + b$$



How do we find b? The same way we did in the previous example: Plug the coordinates of one of the original points in for x and y — either point will do the job, since each of them lies on the line. Let's use the point (–1, 3).

$$\begin{aligned} & y = -2x + b \\ \Rightarrow\quad & \mathbf{3} = -2(\mathbf{-1}) + b \\ \Rightarrow\quad & 3 = 2 + b \\ \Rightarrow\quad & b = 1 \end{aligned}$$

Putting the values of m and b into the formula $y = mx + b$ gives us the line which passes through the two points:

$$\boxed{y = -2x + 1}$$

✓ **Check:** Let's check our final answer. If $y = -2x + 1$ is really the line passing through the two given points, then obviously each point should lie on the line. That is, each point should satisfy the equation of the line.

$y = -2x + 1$ Is this the right equation?

(–1, 3): $\mathbf{3} = -2(\mathbf{-1}) + 1 \Rightarrow 3 = 2 + 1 \Rightarrow 3 = 3$ ✓

(8, –15): $\mathbf{-15} = -2(\mathbf{8}) + 1 \Rightarrow -15 = -16 + 1 \Rightarrow -15 = -15$ ✓

Homework

2. Find the equation of the line passing through the two given points. Also be sure you know how to check your answer:

a. (3, 13) and (–1, 5)
b. (1, –9) and (–5, 39)
c. (0, –1) and (2, 9)
d. (1, –11) and (6, –1)
e. (1, –8) and (–2, 31)
f. (0, 0) and (–5, 35)
g. (7, 7) and (–3, –3)
h. (0, 17) and (–17, 0)
i. (–5, –4) and (2, –11)
j. (1, –3) and (–4, –38)
k. (2, –30) and (–1, –3)
l. (–4, –23) and (–1, 7)
m. (25, –43) and (–10, 27)
n. (1, –13) and (–13, –97)

EXAMPLE 4: **Find the equation of the line passing through the points (2, 5) and (–1, –3).**

Solution: This is the same type of question as in the previous example, except that fractions will be involved. In order to utilize the slope-intercept form of a line, $y = mx + b$, we need to know the slope and the y-intercept. Since neither one of these was explicitly given in the problem, we'll have to calculate them ourselves.

First we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{5-(-3)}{2-(-1)} = \frac{5+3}{2+1} = \frac{8}{3}$$

Our line can now be written as $y = \frac{8}{3}x + b$.

Second we find the value of b. To accomplish this, we can place either point from the problem into the equation we just wrote — let's choose (2, 5):

$$5 = \frac{8}{3}(2) + b \Rightarrow 5 = \frac{16}{3} + b \Rightarrow \frac{15}{3} - \frac{16}{3} = b \Rightarrow b = -\frac{1}{3}$$

Putting it all together, the equation of the line is

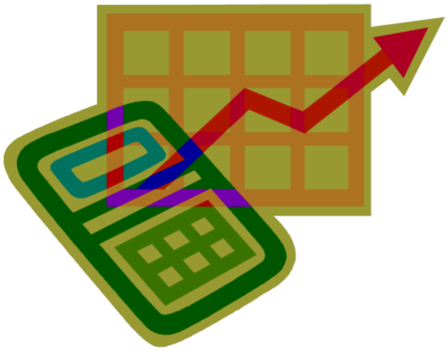
$$\boxed{y = \frac{8}{3}x - \frac{1}{3}}$$

Homework

3. Find the equation of the line passing through the two given points:

a. (10, 9) and (7, –7)	b. (–4, –5) and (–8, 0)
c. (–2, 4) and (1, –6)	d. (–2, 10) and (–7, –7)
e. (7, 5) and (–4, –3)	f. (–8, 4) and (–2, –4)

Review Problems



4. Find the slope and y-intercept of the line $-28x + 4y = 16$ by converting the line to $y = mx + b$ form.

5. Find the equation of the line which has slope 5 and which passes through the point (6, 30).

6. Find the equation of the line which passes through the points (2, 15) and (–2, –9).

7. Find the slope and y-intercept of the line $-45x - 5y = -15$ by converting the line to $y = mx + b$ form.

8. Find the equation of the line which has slope –4 and which passes through the point (–5, 22).

9. Find the equation of the line which passes through the points (–2, –6) and (7, 3).

Solutions

1. a. $y = -3x + 8$ b. $y = 4x - 7$ c. $y = 2x - 10$
d. $y = -3x + 7$ e. $y = 9x$ f. $y = -x + 8$
g. $y = x$ h. $y = 7x + 10$ i. $y = -8x + 13$
j. $y = x - 99$

2. a. $y = 2x + 7$ b. $y = -8x - 1$ c. $y = 5x - 1$
d. $y = 2x - 13$ e. $y = -13x + 5$ f. $y = -7x$
g. $y = x$ h. $y = x + 17$ i. $y = -x - 9$
j. $y = 7x - 10$ k. $y = -9x - 12$ l. $y = 10x + 17$
m. $y = -2x + 7$ n. $y = 6x - 19$

3. a. $y = \frac{16}{3}x - \frac{133}{3}$ b. $y = -\frac{5}{4}x - 10$
c. $y = -\frac{10}{3}x - \frac{8}{3}$ d. $y = \frac{17}{5}x + \frac{84}{5}$
e. $y = \frac{8}{11}x - \frac{1}{11}$ f. $y = -\frac{4}{3}x - \frac{20}{3}$

4. $m = 7$; y-int: $(0, 4)$

5. $y = 5x$

6. $y = 6x + 3$

7. $m = -9$; y-int: $(0, 3)$

8. $y = -4x + 2$

9. $y = x - 4$

> "I haven't failed, I've found 1,000 ways that don't work."

Thomas Edison

CH XX – EQUATION OF A LINE, $y - y_1 = m(x - x_1)$

The slope-intercept form of a line, $y = mx + b$, is perfect when you have the slope and the y-intercept. But the odds of this are slim. It's more likely that you'll be working with the slope and some point on the line other than the y-intercept. So, in the following formula, m is the slope (as before), while (x_1, y_1) represents any given point on the line.

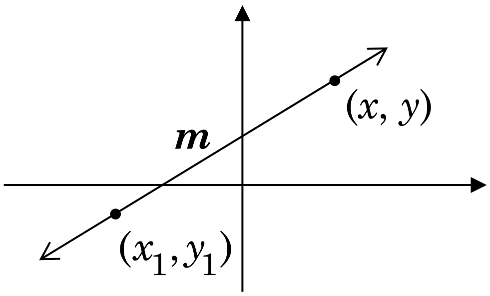
☐ *THE POINT-SLOPE FORMULA*

THEOREM: **The equation of the line with slope m and passing through the point (x_1, y_1) is**

$$\boxed{y - y_1 = m(x - x_1)}$$

The Point-Slope Form of a Line

PROOF: We begin by sketching the line and labeling the given point (x_1, y_1), the slope m, and a *generic point* (x, y):



On the one hand, the slope of the line is given by m. On the other hand, the slope of the line can be calculated using the two points (x, y) and (x_1, y_1): $\frac{y - y_1}{x - x_1}$. And, of course, these two slopes must be the same (since there's only one line involved):

This is the form of a line that is used in Calculus.

$$\frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow \quad \boxed{y - y_1 = m(x - x_1)}$$

Q.E.D.

☐ *BONUS DERIVATION*

Using the point-slope form of a line above, we can derive the slope-intercept form of a line: $y = mx + b$. Here's how:

We assume that the slope of a line is given by m, and that the y-intercept is $(0, b)$, which is simply a point on the line; so it's the point (x_1, y_1) in the formula $y - y_1 = m(x - x_1)$. Thus,

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - b = m(x - 0)$$

$$\Rightarrow \quad y - b = mx$$

$$\Rightarrow \quad y = mx + b$$

and we're done!

☐ *EXAMPLES OF THE POINT-SLOPE FORMULA*

EXAMPLE 1: **Find the equation of the line whose slope is –3 and which passes through the point (8, –2).**

Solution: This is precisely the data we need to use the point-slope form: $y - y_1 = m(x - x_1)$. We're given the slope, so $m = -3$. We're also given a point on the line, so $(x_1, y_1) = (8, -2)$. Plugging these values into the point-slope form gives us

$$y - (-2) = -3(x - 8), \text{ or}$$

$$\boxed{y + 2 = -3(x - 8)}$$

EXAMPLE 2: **Find the equation of the line passing through the two points (3, –5) and (–2, –8).**

Solution: The point-slope form, $y - y_1 = m(x - x_1)$, requires a point (we have two of them), and the slope, which we'll have to calculate ourselves:

$$m = \frac{\Delta y}{\Delta x} = \frac{-5-(-8)}{3-(-2)} = \frac{-5+8}{3+2} = \frac{3}{5}$$

Now, using the point (3, –5) (either point would work), we get our equation

$$y - (-5) = \frac{3}{5}(x - 3), \text{ or } \boxed{y + 5 = \frac{3}{5}(x - 3)}$$

Homework

1. Use the point-slope formula to find the equation of the line with slope 7 and passing through the point (6, –8).

2. Use the point-slope formula to find the equation of the line with slope 0 and passing through the point (–17, 9).

3. Use the point-slope formula to find the equation of the line with slope $-\frac{4}{7}$ and passing through the point $(\frac{1}{2}, \pi)$.

4. Use the point-slope formula to find the equation of the line which passes through the points (–2, 4) and (5, –5).

5. Use the point-slope formula to find the equation of the line which passes through the points $(\pi, \sqrt{2})$ and (–3, 1).

Solutions

1. $y + 8 = 7(x - 6)$

2. $y - 9 = 0$

3. $y - \pi = -\frac{4}{7}\left(x - \frac{1}{2}\right)$

4. $y - 4 = -\frac{9}{7}(x + 2)$

5. $y - \sqrt{2} = \frac{1 - \sqrt{2}}{-3 - \pi}(x - \pi)$ The slope can also be written: $\frac{\sqrt{2} - 1}{\pi + 3}$.

An investment in knowledge

always pays the best interest.

Benjamin Franklin

CH XX – LOG EQUATIONS

Decibel Scale

Now that we have the laws of logs at our disposal, we can solve equations containing logs. We will also revisit the Richter scale, the pH scale, and the decibel scale.

☐ REVIEW OF THE LOG FUNCTION AND LAWS OF LOGS

pH Scale

Richter Scale

Some log equations can be solved by inspection. For instance, the solution of the equation $\log x = 2$ is $x = 100$, since log 100 = 2. But some others aren't so easy. The key to solving log equations will be the conversion from a log expression to an exponential expression; this conversion is none other than our definition of log:

$$\boxed{\log_b x \;=\; y \quad \text{means} \quad b^y \;=\; x}$$

In addition, let's restate the 1st and 2nd laws of logs — these will also be used in this chapter:

First Law of Logs: $\log_b(xy) \;=\; \log_b x + \log_b y$

Second Law of Logs: $\log_b\left(\dfrac{x}{y}\right) \;=\; \log_b x - \log_b y$

☐ LEVEL ONE EQUATIONS

EXAMPLE 1: **Solve for x: $\log_8 x - 2 = 0$**

Solution:

$\log_8 x - 2 = 0$ (the original equation)

$\Rightarrow$ $\log_8 x = 2$ (begin to isolate the x)

$\Rightarrow$ $8^2 = x$ (**change to exponential form**)

$\Rightarrow$ $\boxed{x = 64}$ (calculate the value of x)

EXAMPLE 2: **Solve for y: $\log(3y + 5) = 1$**

Solution: Note: This is the common log, base 10.

$\log(3y + 5) = 1$ (the original equation)

$\Rightarrow$ $10^1 = 3y + 5$ (**change to exponential form**)

$\Rightarrow$ $3y + 5 = 10$ (rearrange and simplify)

$\Rightarrow$ $3y = 5$ (subtract 5 from each side)

$\Rightarrow$ $\boxed{y = \frac{5}{3}}$ (divide each side by 3)

EXAMPLE 3: **Solve for x: $\ln(7 - 4x) = \frac{1}{2}$**

Solution: ***ln*** is just another log, and its base is understood to be ***e***.

$\ln(7 - 4x) = \frac{1}{2}$ (the original equation)

$\Rightarrow$ $e^{1/2} = 7 - 4x$ (**convert to exponent form**)

$$\Rightarrow \quad -4x+7 = \sqrt{e} \qquad \text{(flip the equation around)}$$

$$\Rightarrow \quad -4x = \sqrt{e}-7 \qquad \text{(subtract 7 from each side)}$$

$$\Rightarrow \quad x = \frac{\sqrt{e}-7}{-4} \qquad \text{(divide each side by }-4\text{)}$$

$$\Rightarrow \quad x = \frac{7-\sqrt{e}}{4} \qquad \text{(multiply top and bottom by }-1\text{)}$$

So the exact solution is

$$\boxed{x = \frac{7-\sqrt{e}}{4}}$$

We can use a calculator to approximate the solution as **1.338**.

Homework

1. Solve each log equation:

a. $\log_6 x = 2$

b. $\log_5 x = 3$

c. $\log_2 x = \frac{1}{2}$

d. $\log_3 x = -4$

e. $\log_7(x+2) = 1$

f. $\log(2x-1) = 2$

g. $\ln x = 3$

h. $\ln(x-1) = 1$

i. $\ln(3x-5) = 0$

j. $\ln(2x+5) = \frac{1}{4}$

❑ LEVEL TWO EQUATIONS

EXAMPLE 4: **Solve for x: $\log x + \log 2 = 3$**

Solution: Be honest — you don't like logs, and this equation has two of them in it! But we have a way out, using the first law of logs: $\log(ab) = \log a + \log b$

Let's begin:

$$
\begin{aligned}
& \log x + \log 2 = 3 && \text{(the original equation)} \\
\Rightarrow\ & \log(2x) = 3 && \textbf{(first law of logs)} \\
\Rightarrow\ & 10^3 = 2x && \text{(exponent form -- base is 10)} \\
\Rightarrow\ & 2x = 1000 && \text{(rearrange and simplify)} \\
\Rightarrow\ & \boxed{x = 500} && \text{(divide each side by 2)}
\end{aligned}
$$

EXAMPLE 5: **Solve for n: $\log(n - 3) = \log 5 + 4$**

Solution: For this problem, the second law of logs, $\log\frac{a}{b} = \log a - \log b$, comes into play:

$$
\begin{aligned}
& \log(n-3) = \log 5 + 4 && \text{(the given equation)} \\
\Rightarrow\ & \log(n-3) - \log 5 = 4 && \text{(bring the logs together)} \\
\Rightarrow\ & \log\left(\frac{n-3}{5}\right) = 4 && \textbf{(second law of logs)} \\
\Rightarrow\ & 10^4 = \frac{n-3}{5} && \text{(convert to exponent form)} \\
\Rightarrow\ & n - 3 = 5(10^4) = 50{,}000 && \text{(multiply each side by 5)} \\
\Rightarrow\ & \boxed{n = 50{,}003} && \text{(solve for } n\text{)}
\end{aligned}
$$

As a quick check, note that if we plug 50,003 into the original equation, we are taking the log of a positive number – a good sign, since the log of a negative number or zero is undefined. Using a calculator, we can compute the following:

$\log (n - 3) = \log 5 + 4$

$\log (\mathbf{50{,}003} - 3) = \log 5 + 4$

$\log (50{,}000) = \log 5 + 4$

$4.69897 = 0.69897 + 4$

$4.69897 = 4.69897$ ✓

EXAMPLE 6: **Solve for z: $\log z + \log (z + 15) = 2$**

Solution: Let's just do it:

$\log z + \log (z + 15) = 2$ (the given equation)

$\Rightarrow \log [z(z + 15)] = 2$ (**the first law of logs**)

$\Rightarrow \log (z^2 + 15z) = 2$ (distribute)

$\Rightarrow 10^2 = z^2 + 15z$ (change to exponent form)

$\Rightarrow z^2 + 15z - 100 = 0$ (put in quadratic form)

$\Rightarrow (z + 20)(z - 5) = 0$ (factor)

$\Rightarrow z = -20$ OR $z = 5$ (set each factor to 0)

That yields two potential solutions, but only checking will tell which, if either, will work in the original equation:

$z = -20$: Immediate meltdown:

$\log z + \log (z + 15) = 2$

$\log (-20)$ + . . . we can stop here;

this log is undefined; thus, –20 fails.

z = 5: It might work. Using a calculator:

$$\log z + \log(z+15) = 2$$
$$\log 5 + \log(5+15) = 2$$
$$\log 5 + \log 20 = 2$$
$$0.698970004 + 1.301029996 = 2$$
$$2 = 2 \quad ✓$$

Thus, the final solution to the equation is $\boxed{z = 5}$

EXAMPLE 7: **Solve for x: $\log_2(x+5) = 3+\log_2(x-2)$**

Solution: Notice that the base of the log is 2, but the equation is really no more difficult than if it had a base of 10.

$$\log_2(x+5) = 3+\log_2(x-2) \quad \text{(the original equation)}$$
$$\Rightarrow \quad \log_2(x+5)-\log_2(x-2) = 3 \quad \text{(bring the logs together)}$$
$$\Rightarrow \quad \log_2\left(\frac{x+5}{x-2}\right) = 3 \quad \text{(the second law of logs)}$$
$$\Rightarrow \quad 2^3 = \frac{x+5}{x-2} \quad \text{(change to exponent form)}$$
$$\Rightarrow \quad 8x - 16 = x + 5 \quad \text{(clear the fraction)}$$
$$\Rightarrow \quad 7x = 21 \quad \text{(subtract } x \text{ and add 16)}$$
$$\Rightarrow \quad \boxed{x = 3}$$

To check our candidate, $x = 3$, we see that we won't be able to use our calculator, since log base 2 is not on the machine:

$$\log_2(x+5) \overset{?}{=} 3+\log_2(x-2)$$
$$\log_2(\boxed{3}+5) \overset{?}{=} 3+\log_2(\boxed{3}-2)$$
$$\log_2 8 \overset{?}{=} 3+\log_2 1$$
$$3 \overset{?}{=} 3+0$$
$$3 = 3 \quad ✓$$

EXAMPLE 8: **Solve for x: $\ln x^2 + \ln x + \ln 7 = 3$**

Solution: For this log equation, notice that the log has the understood base of *e*.

$\ln x^2 + \ln x + \ln 7 = 3$

$\Rightarrow \ln (x^2 \cdot x \cdot 7) = 3$ (**sum of logs = log of product**)

$\Rightarrow \ln (7x^3) = 3$ (simple algebra)

$\Rightarrow e^3 = 7x^3$ (**change to exponent form**)

$\Rightarrow x^3 = \dfrac{e^3}{7}$ (divide each side by 7)

$\Rightarrow x = \sqrt[3]{\dfrac{e^3}{7}}$ (take the cube root of each side)

$\Rightarrow x = \dfrac{\sqrt[3]{e^3}}{\sqrt[3]{7}}$ (split the radical)

$\Rightarrow \boxed{x = \dfrac{e}{\sqrt[3]{7}}}$ (simplify the top)

An approximate answer, given by the calculator, is **1.421**.

EXAMPLE 9: **Solve for a: $\ln a = \ln (a + 5) - 4$**

Solution: $\ln a = \ln (a + 5) - 4$ (the original equation)

$\Rightarrow \ln a - \ln (a + 5) = -4$ (subtract $\ln (a + 5)$)

$\Rightarrow \ln\left(\dfrac{a}{a+5}\right) = -4$ (**second law of logs**)

$\Rightarrow \dfrac{a}{a+5} = e^{-4}$ (**convert to exponent form**)

$\Rightarrow \dfrac{a}{a+5} = \dfrac{1}{e^4}$ (convert exponent to fraction)

$\Rightarrow \quad ae^4 = a + 5$ (cross-multiply)

$\Rightarrow \quad ae^4 - a = 5$ (subtract a from each side)

$\Rightarrow \quad a(e^4 - 1) = 5$ (factor out the variable)

$\Rightarrow \quad \boxed{a = \frac{5}{e^4 - 1}}$ (divide each side by $e^4 - 1$)

Homework

Solve each log equation:

2. $\log y + \log 3 = 4$
3. $\ln z = 1 + \ln 5$
4. $\log_3(w+1) - \log_3 w = 3$
5. $\log u + \log (u + 246) = 3$
6. $\log (2x + 1) = 1 - \log (3x - 4)$
7. $\log (2x + 1) - \log 5 = \log x$
8. $\ln (x + 2) - \ln (x - 3) = 2$
9. $\ln x - \ln (x + 5) = 4$
10. $\log_2 x^2 + \log_2 x + \log_2 3 = 5$

11. Solve each log equation:

    a. $\ln x + \ln 5 = 2$
    b. $\log a = 2 + \log 3$
    c. $\log_2(y+1) - \log_2 y = 4$
    d. $\log(u+24) + \log(u+3) = 2$
    e. $\log_5(8x+1) = 2 - \log_5(x-2)$
    f. $\ln(2x) - \ln(x-1) = 4$
    g. $\log_3 n^2 + \log_3 n + \log_3 7 = 2$
    h. $\log(5-x) + \log(x+15) = 2$

12. Check your solution to problem 11h.

☐ LOG APPLICATIONS REVISITED

Now that we can solve equations containing logs, we will go back to our formulas for pH, decibels, and Richter numbers and solve more problems. Here are the three formulas from previous chapters:

$\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles/liter

$D = 10\log\left(\frac{I}{10^{-12}}\right)$ where I is the intensity in watts/m^2

$M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$ where E is the energy in joules

EXAMPLE 10: **If the pH of an acid is 2.3, find the hydrogen ion concentration.**

Solution: To simplify the symbols, let's use just the letter H to represent the concentration.

$$\text{pH} = -\log H$$

$$\Rightarrow \quad 2.3 = -\log H$$

$$\Rightarrow \quad \log H = -2.3$$

$\Rightarrow \quad H = 10^{-2.3}$, and so the concentration is approximately

$$\boxed{0.005 \text{ moles/liter}}$$

EXAMPLE 11: **A certain sound has a decibel value of 85. Find the intensity of the sound.**

Solution: $D = 10\log\left(\dfrac{I}{10^{-12}}\right)$

$$\Rightarrow \quad 85 = 10\log\left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow \quad 8.5 = \log\left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow \quad \frac{I}{10^{-12}} = 10^{8.5}$$

$$\Rightarrow \quad I = 10^{-12}\times 10^{8.5}$$

$$\Rightarrow \quad I = 10^{-3.5}$$

$$\Rightarrow \quad \boxed{I = .00032 \text{ W/m}^2}$$

EXAMPLE 12: **An earthquake had a Richter magnitude of 6. Find the energy release in joules.**

Solution:

$$M = \frac{2}{3}\log\left(\frac{E}{2.5\times 10^4}\right)$$

$$\Rightarrow \quad 6 = \frac{2}{3}\log\left(\frac{E}{2.5\times 10^4}\right) \qquad \text{(we need to solve for } E\text{)}$$

$$\Rightarrow \quad 9 = \log\left(\frac{E}{2.5\times 10^4}\right) \qquad \text{(multiply each side by } \tfrac{3}{2}\text{)}$$

$$\Rightarrow \quad \frac{E}{2.5\times 10^4} = 10^9 \qquad \text{(definition of log)}$$

$$\Rightarrow \quad E = 10^9\times 2.5\times 10^4 \Rightarrow \boxed{E = 2.5\times 10^{13}\text{ J}}$$

Homework

13. The pH of a strong base is 12.3. Find the hydrogen ion concentration.

14. Find the intensity of a sound wave whose decibel value is 50.

15. Calculate the energy released in an earthquake of magnitude 4.5 on the Richter scale.

16. The pH of a weak acid is 5.8. Find the hydrogen ion concentration.

17. Find the intensity of a sound wave whose decibel value is 100.

18. Calculate the energy released in an earthquake of magnitude 7.8 on the Richter scale.

Review Problems

19. Solve for x: $\log_3(x+4) - \log_3 x = 4$

20. Solve for x: $\ln x^3 + \ln x^2 - \ln x - 2 = 0$

21. Solve for x: $\log(x-3) = \log x - 3$

22. Solve for x: $\log_2(x+14) = 5 - \log_2 x$

23. a. Find the hydrogen ion concentration of a base with pH = 12.

b. Find the intensity of a sound whose decibel level is 40.

c. Find the energy released in an earthquake of Richter 3.

24. True/False:

a. The equation $\log_2 x = 0$ has a solution.

b. The solution of $\ln(7-4x) = \frac{1}{2}$ is $x = \frac{1}{14-8x}$.

c. The solution of $\log x + \log 5 = 4$ is $x = 2000$.

d. The solution of $\log_3(x+1) - \log_3(x-1) = 3$ is $x = 14/13$.

e. The equation $\log n = 2 - \log(n+15)$ has two solutions.

f. The solution of $\ln x^3 + \ln x^2 + \ln x + \ln 2 = 10$ is $x = \sqrt[6]{\frac{e^{10}}{2}}$.

g. If an acid has a pH of 2.5, then its hydrogen ion concentration is 0.02 moles/liter.

h. The intensity of a sound whose decibel level is 125 is 3.1623 W/m^2.

i. If the Richter magnitude of an earthquake is 9, then the energy release is 7.9057×10^{17} joules.

Solutions

1. a. 36 b. 125 c. $\sqrt{2}$ d. $\frac{1}{81}$ e. 5

f. $\frac{101}{2}$ g. e^3 h. $e + 1$ i. 2 j. $\frac{\sqrt[4]{e} - 5}{2}$

2. $\frac{10{,}000}{3}$ **3.** $5e$, which is ≈ 13.5914 **4.** $\frac{1}{26}$

5. –250 is an extraneous solution; the solution is $\{4\}$.

6. You should reach the quadratic equation $6x^2 - 5x - 14 = 0$, whose two solutions are 2 and $-\frac{7}{6}$. Only 2 works.

7. $\frac{1}{3}$

8. $\ln(x+2) - \ln(x-3) = 2 \Rightarrow \ln\frac{x+2}{x-3} = 2 \Rightarrow \frac{x+2}{x-3} = e^2$

$\Rightarrow x + 2 = e^2x - 3e^2 \Rightarrow x - e^2x = -3e^2 - 2$

$\Rightarrow x(1-e^2) = -3e^2 - 2 \Rightarrow x = \frac{-3e^2-2}{1-e^2}$, or $\frac{3e^2+2}{e^2-1}$

9. $x = \frac{5e^4}{1-e^4}$ is the solution obtained by following the rules. The problem is that this value of x is negative. Placing this value of x into the original equation yields an error; the final solution is therefore ∅, a notation that indicates No Solution.

10. You should get the equation $3x^3 = 32$.

Solving for x gives $\sqrt[3]{\frac{32}{3}}$ or $2\sqrt[3]{\frac{4}{3}}$ or $\frac{2\sqrt[3]{36}}{3}$.

11. a. $\frac{e^2}{5}$ b. 300 c. $\frac{1}{15}$ d. 1

e. 3 f. $\frac{e^4}{e^4-2}$ g. $\sqrt[3]{\frac{9}{7}}$ h. –5

12. We suppose that $x = -5$ is a solution of $\log(5-x) + \log(x+15) = 2$. Placing –5 in for x in the original equation, $\log(5-x) + \log(x+15) = 2$, gives

$$\log(5-(-\mathbf{5})) + \log(-\mathbf{5}+15) = \log(5+5) + \log(-5+15)$$
$$= \log 10 + \log 10 = 1 + 1 = \mathbf{2}$$

It works. ☺

13. 5.01×10^{-13} moles/liter

14. 10^{-7} W/m^2

15. 1.41×10^{11} joules

16. 0.000001585 moles/liter

17. 0.01 W/m^2

18. 1.253×10^{16} J

19. $x = \frac{1}{20}$

20. $x = \sqrt{e}$

21. $x = \frac{1000}{333}$

22. $x = 2$

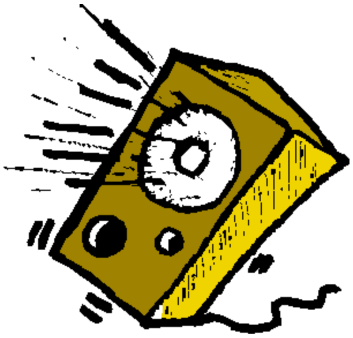
23. a. 1×10^{-12} moles/liter b. 1×10^{-8} W/m^2 c. 7.91×10^{8} joules

24. a. T b. F c. T d. T e. F f. T g. F h. T i. T

Education is what survives when what has been learned has been forgotten.

BF Skinner

CH NN – LOG FUNCTIONS



The **decibel scale** for measuring the loudness of sound is discussed at the end of the chapter.

To better understand logs, it would be nice to graph them; but to do that, we need to express the log concept as a *function*.

You learned how to perform operations like *squaring* back in basic algebra. But it wasn't until later in your studies that you looked at squaring as a function: a collection of inputs and outputs where the output is always the *square* of the input, and where each input has exactly one output. This new function approach led to concepts such as domain, notation like $f(x) = x^2$, and the parabolic graphs of the squaring functions.

We do the same thing now with *logs*. We've learned what the definition of log is, and we're able to calculate logs, using a variety of bases. Now it's time to formalize the concept of log in the parlance of functions.

❑ DEFINITION

Let ***b*** be a positive real number not equal to 1. Then the function

$$y = \log_b x$$

is called the ***log base b*** function.

Note that another way to describe what the base ***b*** can be is to say that ***b*** must be in the interval $(0, \infty) - \{1\}$.

As with exponential functions, the base, *b*, of a logarithmic function satisfies the property:

$b > 0,\ b \neq 1$

Homework

1. In the log function $y = \log_b x$, what real numbers can ***b*** be?

2. Let $f(x) = \log_2 x$. Compute:

   a. $f(2)$ b. $f(4)$ c. $f(16)$ d. $f(128)$ e. $f(1)$
   f. $f(\frac{1}{2})$ g. $f(\frac{1}{8})$ h. $f(0)$ i. $f(-1)$ j. $f(-2)$

3. Let $g(x) = \log x$. Compute:

   a. $g(10)$ b. $g(1)$ c. $g(0)$
   d. $g(100)$ e. $g(-100)$ f. $g(\frac{1}{1000})$

4. Let $h(x) = \ln x$. Compute:

   a. $h(0)$ b. $h(1)$ c. $h(e)$ d. $h(e^2)$ e. $h(e^{10})$
   f. $h(e^x)$ g. $h(\frac{1}{e})$ h. $h(\sqrt{e})$ i. $h\left(\frac{1}{\sqrt{e}}\right)$ j. $h(-e)$

Note that there are a variety of *log* functions, one for each valid value of the base, ***b***.

❑ DOMAIN

From the previous Homework we extract three problems:

$\log_2\left(\frac{1}{8}\right) = -3$ $\log 0$ is Undefined $\ln(-e)$ is Undefined

NOTE: What these three logs, along with the rest of the Homework, should tell you is that the log function (for any legal base) is defined only for inputs that are positive numbers; that is, you can take the log of positive numbers only. Equivalently, you can never take the log of 0 or any negative number. In short,

The domain of the function $y = \log_b x$ is $\boldsymbol{x > 0}$.

[In interval notation, the domain is **(0, ∞)**.]

EXAMPLE 1: **Find the domain of each function:**

A. $f(x) = \log_3(x+7)$

We can take the log of positive quantities only; therefore, we must solve the inequality $x + 7 > 0$, which implies that $x > -7$. So the domain is

$$\boxed{x > -7}$$

Interval notation: **(–7, ∞)**

B. $g(x) = \log(x^2 - 5x - 14)$

Again, we can take the log of positive quantities only. So we must solve the inequality $x^2 - 5x - 14 > 0$. Using the *Boundary Point Method*, we solve the associated equation and calculate the boundary points: $x^2 - 5x - 14 = 0 \Rightarrow$ $(x + 2)(x - 7) = 0 \Rightarrow$ -2 and 7 are the boundary points.

By choosing test points and checking them in the original inequality, together with checking the boundary points, you can verify that the domain is

$$\boxed{x < -2 \text{ OR } x > 7}$$

Interval notation:

$(-\infty, -2) \cup (7, \infty)$

C. $h(x) = \ln(25 - x^2)$

To solve the inequality $25 - x^2 > 0$, we solve the equation to get the boundary points -5 and 5. When the test points and boundary points are tested, the domain is

$$\boxed{-5 < x < 5 \qquad \text{Interval notation: } \mathbf{(-5, 5)}}$$

Homework

5. Consider the function $y = \ln(x^2 - 3x - 10)$. Without actually calculating the domain, determine which of the following are in the domain of the function:

   a. π b. -2 c. -10 d. 0 e. 5 f. e g. 12

6. Find the domain of each function:

   a. $y = \log_{23} x$
   b. $y = \log(8 - 2x)$
   c. $y = \ln(x^2 + 2x - 3)$
   d. $y = \log_5(x^2 - 49)$
   e. $y = \log_{12}(x^2 + 1)$
   f. $y = \log_\pi(100 - x^2)$

☐ GRAPHING

EXAMPLE 2: **Graph: $y = \log_2 x$**

Solution:

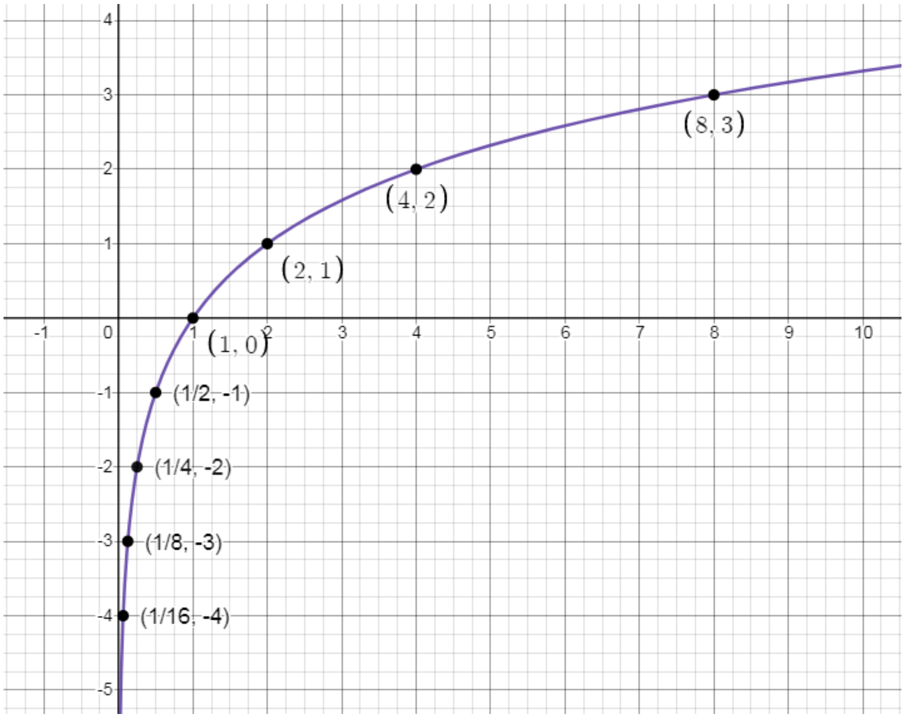
If $x = 8$, then $y = \log_2 8 = 3$ (since $2^3 = 8$).

If $x = 2$, then $y = \log_2 2 = 1$ (since $2^1 = 2$).

If $x = 1/2$, then $y = \log_2 1/2 = -1$ (since $2^{-1} = 1/2$)

Continuing in this way, and remembering that the domain of the log function is $x > 0$, we can calculate the following ordered pairs in the function. Plotting the points in the table and connecting them with a smooth curve gives us the following graph:

x	y
1	0
2	1
4	2
8	3
16	4
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3
$\frac{1}{16}$	-4



Let's confirm what we said earlier. The graph seems to indicate that the **domain** of the function is $\boldsymbol{x > 0}$; that is, x must be a positive real number.

As for **intercepts**, the table and the graph indicate that (1, 0) is the only x-intercept. There is no y-intercept, since x can never be 0.

Let's look at some **limits** for this function. Check out the first five points in the x-y table. As x grows larger and larger, so does y, albeit rather slowly. As slowly as the graph rises, it nevertheless rises higher and higher as x gets larger (tough to see; just trust me). Therefore,

As $\boldsymbol{x \to \infty, y \to \infty}$,

which can be written $\lim\limits_{x \to \infty} \log_2 x = \infty$.

Now look at how the graph gets closer and closer to the y-axis as x gets closer and closer to 0. Remember, even though x can <u>never be</u> 0, that doesn't prevent us from analyzing what happens to y when x <u>approaches</u> 0. But, of course, looking at the graph (and the domain of the function) shows us that x cannot approach 0 from the left because there's no graph there, so we consider x approaching 0 only from the right.

Now to the question: As x approaches 0 from the right, what does y approach? It's going down and down, so it appears to be approaching $-\infty$. Here's what we're saying:

As $\boldsymbol{x \to 0}$ (from the right)**, $\boldsymbol{y \to -\infty}$**

The two limits discussed seem to give evidence that the y-values take on all real numbers. In addition, the second limit indicates that the function has a **vertical asymptote** at $x = 0$ (the y-axis).

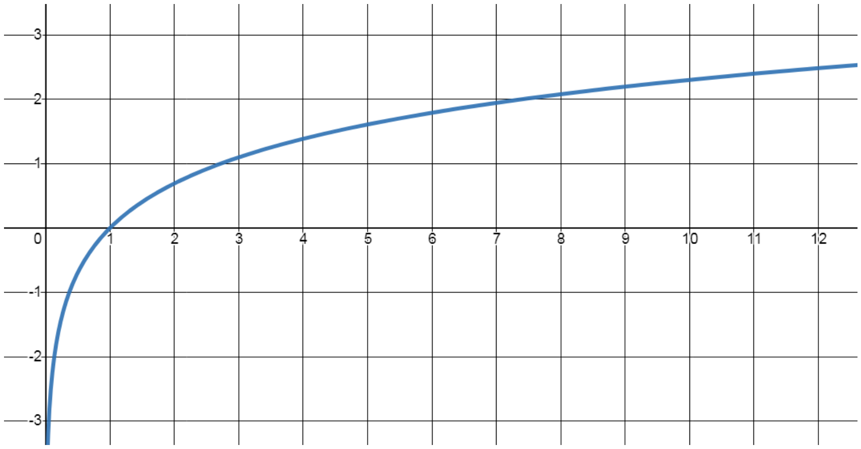
EXAMPLE 3: **Graph:** $y = \ln x$

Solution: Using a calculator when necessary, we can compute the following ordered pairs for the function:

$(\frac{1}{10}, -2.3)$ $(\frac{1}{2}, -.69)$ $(1, 0)$

$(e, 1)$ $(5, 1.61)$ $(8, 2.08)$

These points, together with the conclusions of Example 2, yield the following graph:



EXAMPLE 4: **Graph:** $y = \log(x + 2)$

Solution: Note that this is a common log, base 10. We begin our analysis of this function by finding the **domain**. To do this, we create the inequality $x + 2 > 0$, which implies that $\boldsymbol{x > -2}$.

The ***x*-intercept** is found by setting y to 0. This produces the equation $\log(x + 2) = 0$. Converting this to log form gives $10^0 = x + 2$, from which it follows that $x = -1$. Thus, the x-intercept is **(–1, 0)**.

To calculate the ***y*-intercept**, we of course set x to 0. This yields the equation $y = \log(0 + 2)$, which produces a y-value of about 0.3. Therefore, the y-intercept is **(0, 0.3)**.

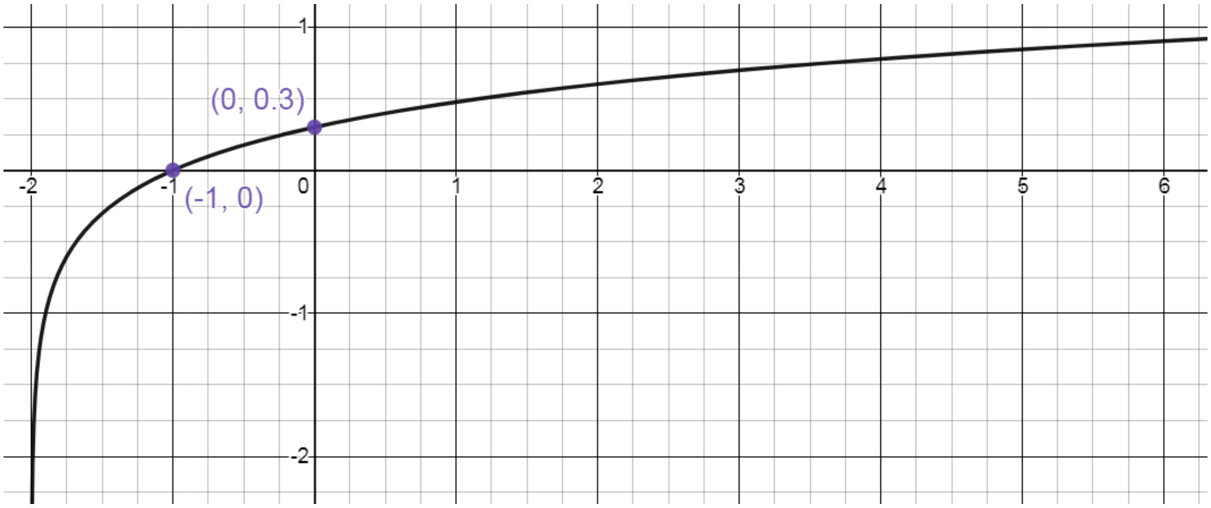
We can find a few ordered pairs for our graph without a calculator by letting $x = -1\frac{9}{10}$, 8, and 98:

$(-1\frac{9}{10}, -1)$ (8, 1) (98, 2)

We can use our calculator to find even more ordered pairs:

(–1.98, –1.7) (3, 0.7) (5, 0.8) (6, 0.9)

[If you're smart, you'll verify each of these ordered pairs.]



We can be pretty sure that the graph goes infinitely down, and we've already determined that log functions go infinitely up, although extremely slowly. From these facts, we see that all y-values are obtained by the function. We can summarize these ideas with a pair of limits:

As $x \to -2$ (from the right)**, $y \to -\infty$**

which can be written $\lim\limits_{x \to -2^+} \log_2(x+2) = -\infty$.

As $x \to \infty$, $y \to \infty$

As a final observation, we can see that the line $x = -2$ is a vertical asymptote, but there is NO horizontal asymptote.

Homework

7. Prove that the graph of $y = \log_2 x$ eventually reaches a height of 10. Hint: Find an x that yields a y of 10.

8. Explain why x cannot approach 0 from the left in the function $f(x) = \log x$.

9. Graph $y = \log(x - 1)$ and describe completely.

10. Graph the three functions $\ln x$, $\ln(x-2)$, and $\ln(x+3)$ on the same grid. Describe how the –2 and the 3 affect the graph of $\ln x$.

11. Find the domain and the vertical asymptote of $y = \ln(x - 5)$.

12. Graph the three functions $\ln x$, $\ln x + 2$, and $\ln x - 1$ on the same grid. Describe how the 2 and –1 affect the graph of $\ln x$. Note: the expression $\ln x + 2$ means take the *ln* first and then add 2.

13. Find the domain and the vertical asymptote of $y = \ln x + 9$.

❑ LOG FUNCTIONS IN A NUTSHELL

$$\underset{\substack{\uparrow\\ \textbf{any real}\\ \textbf{number}}}{y} = \log_{\underset{\substack{\uparrow\\ \textbf{positive,}\\ \neq \mathbf{1}}}{b}} \underset{\substack{\nwarrow\\ \textbf{positive}}}{x} \quad \text{means} \quad x = b^y$$

$$b \in (0, \infty) - \{1\}$$

$$\text{Domain} = (0, \infty)$$

Homework

14. Let $y = \log_b x$. True or False?

a. x must be a non-negative real number.

b. y can be any real number.

c. b could be 1.

d. The domain of the function is $[0, \infty)$.

e. y can be any real number.

f. The equation is equivalent to $b^x = y$.

g. The graph of the function lies in Quadrants I and II.

h. x could be 1.

i. y could be 1.

j. As $x \to \infty$, $y \to 0$.

k. As $x \to -\infty$, $y \to$ nothing at all.

l. As $x \to 0$ (from the left), $y \to -\infty$.

m. As $x \to 1$ (from the right), $y \to 0$.

n. As $x \to 1$ (from the left), $y \to 0$.

o. x must be a positive number.

p. b could be 0.

q. b could be negative.

❑ THE DECIBEL SCALE

Named after Alexander Graham Bell, the **decibel scale** is a measure of the loudness of sound waves. If I is the intensity of the sound, measured in watts per square meter (W/m^2), then the number of decibels is given by

$$D = 10 \log\left(\frac{I}{10^{-12}}\right).$$

[The denominator 10^{-12}, or 0.000000000001, is the intensity of the softest sound the human ear can detect.]

Calculator Hint: On a TI-30, to enter a number like 7.3×10^{13}, first press 7.3, then press the "EE" button, and then press 13. To enter the number 10^{-12}, press 1, then EE, then 12, and then +/–.

EXAMPLE 5: **The sound produced by a jet engine has an intensity of 8.3 × 10^{2} W/m^{2}. Find the decibel value of this sound.**

Solution:

$$\begin{aligned} D &= 10\log\left(\frac{I}{10^{-12}}\right) \\ &= 10\log\left(\frac{8.3\times10^{2}}{10^{-12}}\right) \\ &= 10\log(8.3\times10^{14}) \qquad \text{(bring up the negative exponent)} \\ &= 10(14.919) \\ &= 149 \end{aligned}$$

The jet engine sound has a value of

$$\boxed{149 \text{ decibels}}$$

Homework

15. Find the decibel value of a whisper whose intensity is 5.2×10^{-10} W/m^2.

16. Find the decibel value of each sound whose intensity is given:

    a. 1.0×10^{-12} W/m^2 b. 2.3×10^{-9} W/m^2

    c. 6.3×10^{-7} W/m^2 d. 8.7×10^{-5} W/m^2

Review Problems

17. Describe the legal values of b in the function $f(x) = \log_b x$.

18. a. If $f(x) = \log_2 x$, calculate $f(1024) + f(1/8)$.
    b. If $g(x) = \log x$, calculate $g(1000) - g(0.01)$.
    c. If $h(x) = \ln x$, calculate $h(e^6) + h(1) - h(\sqrt[3]{e})$.
    d. Calculate $f(512) + g(1 \text{ million}) + h(e)$

19. Find the domain: $f(x) = \log(35 - 7x)$

20. Find the domain of $y = \ln(x^2 - 144)$.

21. Prove that the function $y = \log_2(x - 2)$ has no y-intercept.

22. If $y = \log_5(7x - 21)$, find x if $y = 2$.

23. Graph: $y = \ln(x-2)$ State any intercepts and asymptotes.

24. Let $f(x) = \ln x$ and $g(x) = \ln(x-3)+4$. Describe the graph of g relative to the graph of f.

25. Consider the function $y = \ln x$. Find the following **limits**:

    a. As $x \to \infty$, $y \to$ _____

    b. As $x \to -\infty$, $y \to$ _____

    c. As $x \to 0$ (from the right), $y \to$ _____

    d. As $x \to 0$ (from the left), $y \to$ _____

26. Use the formula $D = 10\log\left(\frac{I}{10^{-12}}\right)$ to find the decibel level of a sound whose intensity is 7.2×10^{-5} W/m^2.

27. True/False:

    a. The base of a log function can be any positive real number.
    b. The base of a log function can be any positive real number <u>except 1</u>.
    c. The domain of the log function $y = \log_b x$ is $x > 0$.
    d. If $g(x) = \log_2(x+10)$, then $g(22) = 5$.
    e. The domain of the function $y = \log_6(x^2-25)$ is $x \le -5$ OR $x \ge 5$.
    f. The x-intercept of the graph of $y = \log(x+2)$ is $(-1, 0)$.
    g. The graph of a log function has a horizontal asymptote.
    h. The graph of a log function has a vertical asymptote.
    i. The graphs of $f(x) = \ln(x+3)$ and $g(x) = \ln x + 3$ are identical.
    j. If the intensity of a sound is 2.765×10^3 W/m^2, the decibel value for that sound is about 154.

Solutions

1. b, the base of the log function, can be any real number greater than 0 but not equal to 1. That is, $b > 0$, $b \neq 1$. [This is exactly the same as what b is allowed to be in the exponential function $y = b^x$.]

2. a. 1 b. 2 c. 4 d. 7 e. 0
f. −1 g. −3 h. Undefined i. Undefined j. Undefined

3. a. 1 b. 0 c. Undefined d. 2 e. Undefined f. −3

4. a. Undefined b. 0 c. 1 d. 2 e. 10
f. x g. −1 h. $\frac{1}{2}$ i. $-\frac{1}{2}$ j. Undefined

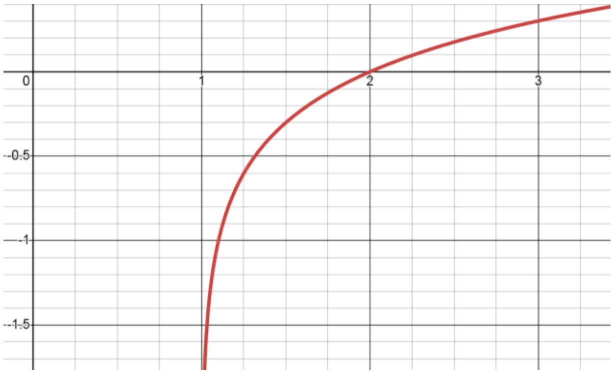
5. c. and g. only, since the quantity $x^2 - 3x - 10$ must be > 0.

6. a. $x > 0$
b. $8 - 2x > 0 \Rightarrow -2x > -8 \Rightarrow x < 4$
c. $x < -3$ OR $x > 1$
d. $x < -7$ OR $x > 7$
e. $\mathbb{R}$
f. $-10 < x < 10$

7. Setting $10 = \log_2 x \quad \Rightarrow \quad x = 2^{10} = 1024$

8. The domain of $\log x$ is $x > 0$. And x can't approach 0 from the left because x can't be negative; that is, there's no graph to the left of the origin.

9.



The domain is all real numbers greater than 1: $x > 1$.

The x-intercept is (2, 0).

There is no y-intercept.

There's a vertical asymptote at $x = 1$.

There is no horizontal asymptote.

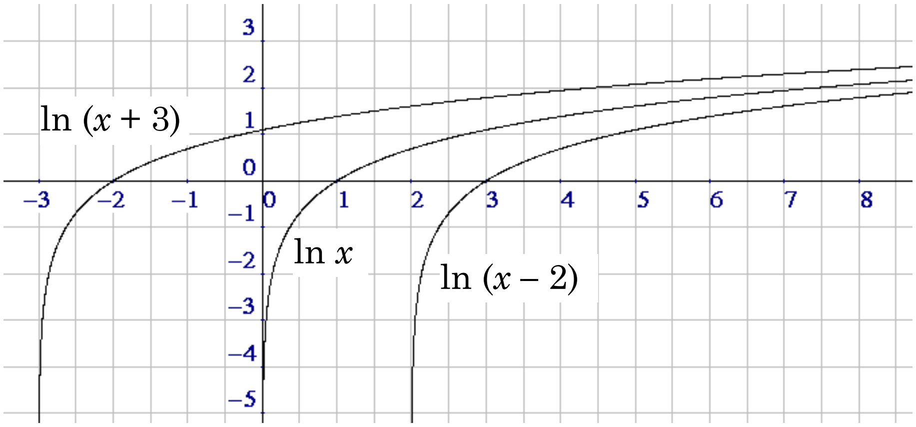
Some important limits:

The limit as $x \to 1$ (from the left) does not exist.

As $x \to 1$ (from the right), $y \to -\infty$.

As $x \to \infty$, $y \to \infty$.

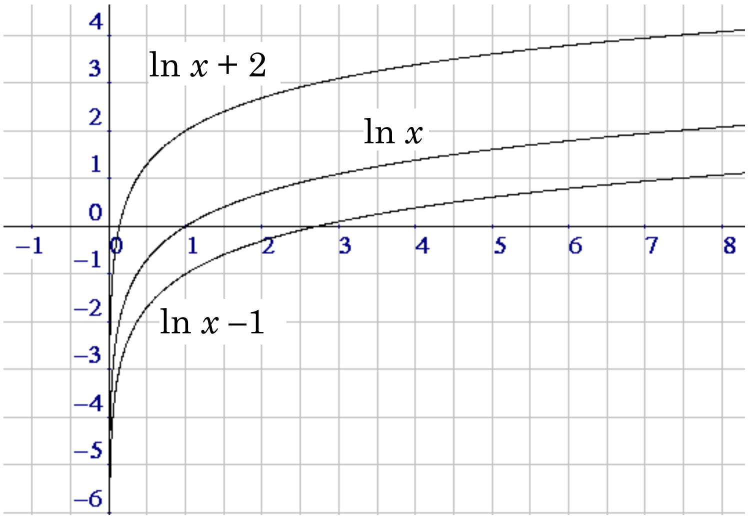
10.



The –2 shifts the graph of $\ln x$ two units to the *right*.
The +3 shifts the graph of $\ln x$ three units to the *left*.

11. Domain: All $x > 5$, or $(5, \infty)$ Vert Asy: $x = 5$

12.



The +2 shifts the graph of ln x two units *up*.
The −1 shifts the graph of ln x one unit *down*.

13. Domain : All $x > 0$, or $(0, \infty)$ Vert Asy: $x = 0$

14. a. F b. T c. F d. F e. T f. F g. F h. T i. T
j. F k. T l. F m. T n. T o. T p. F q. F

15. 27 decibels

16. a. 0 b. 33.6 c. 58 d. 79.4

17. $b > 0$, $b \neq 1$

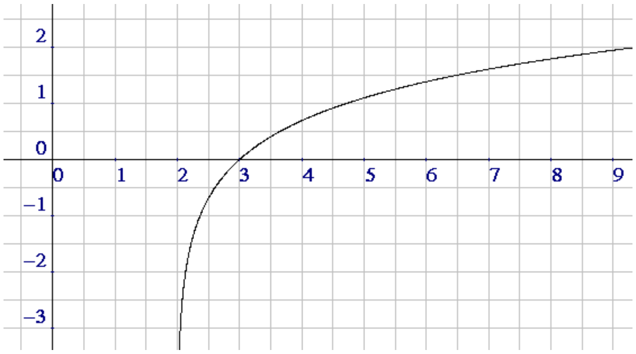
18. a. $10 + (-3) = 7$ b. $3 - (-2) = 5$ c. $6 + 0 - 1/3 = 17/3$
d. $9 + 6 + 1 = 16$

19. $x < 5$ **20.** $x < -12$ OR $x > 12$

21. Setting $x = 0$ yields $\log_2(-2)$, not defined. Therefore, no y-intercept.

22. $2 = \log_5(7x - 21) \Rightarrow 7x - 21 = 5^2 \Rightarrow x = \frac{46}{7}$

23.



x-int: (3, 0)

vert asym: $x = 2$

24. *g* is obtained by moving *f* 3 units to the right and 4 units up.

25. a. ∞ b. Does not exist c. –∞ d. Does not exist

26. 78.6 decibels

27. a. F b. T c. T d. T e. F
f. T g. F h. T i. F j. T

Knowledge is the eye of desire and can become the pilot of the soul.

WILL DURANT, AMERICAN WRITER, HISTORIAN, AND PHILOSOPHER

CH XX – THE LAWS OF LOGS

Since a log is really just an exponent, and since we have laws for exponents, it makes sense that we need to learn some laws for logs, too. These laws of logs will be used to solve new kinds of equations. This chapter also shows us another application of logs, the Richter scale, for measuring the magnitude of an earthquake.

Adapazari, Turkey
Aug 17, 1999
Richter 7.8

☐ *THE TWO CANCELATION RULES*

Carefully study the next five examples:

$$\log_9 9^2 = \log_9 81 = 2$$

$$\log_{10} 10^5 = \log_{10} 100{,}000 = 5$$

$$\log_5 5^3 = \log_5 125 = 3$$

$$\log_2 2^1 = \log_2 2 = 1$$

$$\log_e e^0 = \log_e 1 = 0$$

Something's going on here. In each problem, the final answer matches the exponent at the front of the problem. For example,

$$\log_9 9^{\boxed{2}} = \boxed{2}$$

Also notice that the base of the log matches the base of the expression that we're taking the log of. For instance,

$$\log_{\boxed{10}} \boxed{10}^5 = 5$$

Using these two insights, we might now see that $\log_{12} 12^7 = 7$. We can now generalize this whole discussion into the first of two cancelation rules:

$$\boxed{\log_b b^x = x}$$

We call it a canceling rule because the log function (done second) cancels out the exponential function (done first), leaving just the *x*.

To motivate the second cancelation rule, consider the following four calculations:

$$10^{\log_{10} 1000} = 10^3 = 1000$$

$$2^{\log_2 32} = 2^5 = 32$$

$$5^{\log_5 125} = 5^3 = 125$$

$$7^{\log_7 1} = 7^0 = 1$$

For each example, notice that the base of the entire question matches the base of the log — for example, $\boxed{7}^{\log_{\boxed{7}} 1}$. We also see that in every case, the final answer matches the number we're taking the log of — for instance, $2^{\log_2 \boxed{32}} = 2^5 = \boxed{32}$. In a nutshell,

$$\boxed{b^{\log_b x} = x}$$

This is a canceling rule because the exponential function (done second) cancels out the log function (done first), leaving just the *x*.

EXAMPLE 1: **Use the canceling rules to simplify each expression:**

A. $\log_3 3^n = n$ B. $\log 10^{x+y} = x+y$

C. $7^{\log_7(ab)} = ab$ D. $e^{\ln(e+4)} = e+4$

E. $\log_4 5^n$ cannot be simplified by a canceling rule, since the base of the exponential (the 5) does not match the base of the log (the 4).

F. $8^{\log 7}$ also cannot be simplified by a canceling rule, since the base of the exponential (the 8) does not match the base of the log (the 10).

Homework

1. In the second canceling rule, explain why we must restrict x to the positive real numbers.

2. Simplify each expression:

a. $\log 10$	b. $\ln e$	c. $\log 10^{u+v}$	d. $\ln e^{xyz}$
e. $\log_9 9^{500}$	f. $10^{\log(xy)}$	g. $e^{\ln(\log 7)}$	h. $2^{\log_2(\ln e)}$
i. $3^{\log_3(a-b)}$	j. $\log_2 2^R$	k. $\log e^x$	l. $e^{\log Q}$

3. A common student error is to figure that $\log_b(x+y) = \log_b x + \log_b y$. We need to dispel this myth right now. Let $b = 2$, so that we're working with base 2. Then let both x and y equal 8. Show that the left side of the formula results in 4, whereas the right side comes out 6.

4. Prove that the conjecture log (xy) = (log x)(log y) is false. Hint: Let x = 100 and y = 1000 and work out each side.

☐ THE SUM OF LOGS

We know how to calculate certain logs in our heads; for example, $\log_5 25 = 2$, because 5^2 = 25. Sometimes in a problem we have the sum or difference of two logs, and our goal is to convert that sum or difference into a single log. For instance, here's how we can

Convert $\log 100 + \log 1000$ *into a single common log.*

Start with the sum of the logs: $\log 100 + \log 1000$
Calculate each log separately: $2 \quad + \quad 3$
And add the numbers: 5

Now we have to write 5 as the common log of something.

Since 10^5 = 100,000, we can finish by writing $\log 100{,}000$

In short, $\log 100 + \log 1000 = \log 100{,}000$

For a second example, let's

Find the sum of $\ln e^5$ *and* $\ln e^3$.

Start with the sum of the logs: $\ln e^5 + \ln e^3$
Calculate each log separately: $5 \quad + \quad 3$
And add the numbers: 8
Now we have to write 8 as the natural log of something. $\ln e^8$

That is, $\ln e^5 + \ln e^3 = \ln e^8$

Homework

5. Express each sum of logs as a single log:

a. $\log 10 + \log 1000$

b. $\log 10{,}000 + \log 100$

c. $\log 10 + \log 1$

d. $\log 1000 + \log 1000$

e. $\log_2 8 + \log_2 64$

f. $\log_3 9 + \log_3 3$

g. $\log_4 64 + \log_4 4 + \log_4 1$

h. $\log 10 + \log 100 + \log 1000$

i. $\ln e^5 + \ln e^7$

j. $\ln e^{10} + \ln e^{10}$

☐ *THE DIFFERENCE OF LOGS*

To find the difference of logs, we use the same logic as above:

Express $\log_2 32 - \log_2 8$ *as a <u>single</u> log.*

$\log_2 32 - \log_2 8$

$= 5 - 3$ (since $2^5 = 32$ and $2^3 = 8$)

$= 2$ (arithmetic)

$= \log_2 4$ (since $2^2 = 4$)

In short, $\log_2 32 - \log_2 8 = \log_2 4$

For a second difference of logs problem,

Express $\ln e^9 - \ln e^6$ *as a single log.*

$\ln e^9 - \ln e^6$

$= 9 - 6$ (take the individual logs)

$= 3$ (arithmetic)

$= \ln e^3$ (write 3 as an ln)

So, $\ln e^9 - \ln e^6 = \ln e^3$

Homework

6. Express each difference of logs as a single log:

a. $\log 100{,}000 - \log 1000$	b. $\log 1000 - \log 10$
c. $\log_5 125 - \log_5 25$	d. $\ln e^{10} - \ln e^{7}$
e. $\log_6 36 - \log_6 6$	f. $\ln e^{20} - \ln e^{5}$
g. $\log 1000 - \log 100$	h. $\log_8 512 - \log_8 64$
i. $\log_2 32 - \log_2 8$	j. $\log_{12} 144 - \log_{12} 1$

☐ *SYNOPSIS OF THE TWO PRECEDING SECTIONS*

Let's look at two of the results from the preceding sections:

1) $\log 100 + \log 1000 \ = \ \log 100{,}000$

This shows that the sum of two logs can be combined into a single log — as long as the numbers are multiplied: $100 \times 1000 \ = \ 100{,}000$

2) $\log_2 32 - \log_2 8 \ = \ \log_2 4$

This shows that the difference of two logs can be combined into a single log — as long as the numbers are divided: $\dfrac{32}{8} = 4$

These two results, that the sum of logs is the log of the product, and that the difference of logs is the log of the quotient, can be written like this:

$$\log_b x + \log_b y = \log_b (xy)$$

$$\log_b x - \log_b y = \log_b \frac{x}{y}$$

Traditionally, these two laws of logs are written the other way around; indeed, that's the way we need one of them in the next section.

$$\boxed{\begin{aligned} \log_b (xy) &= \log_b x + \log_b y \\ \log_b \frac{x}{y} &= \log_b x - \log_b y \end{aligned}}$$

The log of a product is the sum of the logs.

The log of a quotient is the difference of the logs.

EXAMPLE 2: **Use the two Laws of Logs to expand each expression:**

A. $\log(ABC) = \log A + \log B + \log C$

B. $\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln z = \ln x + \ln y - \ln z$

C. $\log_2\left(\frac{a}{bc}\right) = \log_2 a - \log_2(bc) = \log_2 a - (\log_2 b + \log_2 c)$

$= \log_2 a - \log_2 b - \log_2 c$

❑ *THE LOG OF A POWER*

Another type of log expression we'll come across is the log of a power, e.g., $\log(7^3)$. We can break it down and use the notion of exponent and the "log of a product" law above to do the problem. Let's work $\log 7^3$:

$$\log 7^3$$

$$= \log(7 \cdot 7 \cdot 7) \quad \text{(definition of exponent)}$$

$$= \log 7 + \log 7 + \log 7 \quad \text{(use the "log of a product" law)}$$

$$= \quad \mathbf{3\log 7} \qquad \text{(combine like terms)}$$

This may not seem very profound, but this maneuver will be necessary to solve exponential equations later in the course. Let's do one more example, and we'll even put a variable into the expression.

$$\ln x^4 = \ln(x \cdot x \cdot x \cdot x) = \ln x + \ln x + \ln x + \ln x = \mathbf{4\ln x}$$

This "log of a power" law can be summarized as

$$\boxed{\log_b x^n = n\log_b x}$$

To calculate the log of a power, bring down the power as a coefficient.

☐ *SUMMARY OF LOGS*

We can summarize our knowledge of **logs** in the following six statements — one definition, two cancelation rules, and three laws:

Statement	Example
$y = \log_b x$ means $b^y = x$	$2 = \log 100$ means $10^2 = 100$
$\log_b b^x = x$	$\ln e^T = T$
$b^{\log_b x} = x$	$2^{\log_2 99} = 99$
$\log_b(xy) = \log_b x + \log_b y$	$\log(100x) = 2 + \log x$
$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\ln \frac{x}{e} = \ln x - 1$
$\log_b x^n = n\log_b x$	$\log_5 \sqrt[7]{Q} = \frac{1}{7}\log_5 Q$

<u>EXAMPLE 3:</u> **Use one of the Laws of Logs to expand each expression:**

A. $\log(7x) = \log 7 + \log x$

B. $\ln(xyz) = \ln x + \ln y + \ln z$

C. $\ln(ex) = \ln e + \ln x = 1 + \ln x$

D. $\log_2 \frac{8}{3} = \log_2 8 - \log_2 3 = 3 - \log_2 3$

E. $\log \frac{A}{B} = \log A - \log B$

F. $\log_{12}[(a+b)(a-b)] = \log_{12}(a+b) + \log_{12}(a-b)$

G. $\log(x^3) = 3\log x$

H. $\log_3 \sqrt{x} = \log_3 x^{1/2} = \frac{1}{2}\log_3 x$

I. $\ln \sqrt[3]{w^2} = \ln w^{2/3} = \frac{2}{3}\ln w$

J. $(\ln x)^2$ cannot be simplified because the power is on the $\ln x$, not the x. Note that $\ln x^2 = 2\ln x$. See the difference?

<u>EXAMPLE 4:</u> **Use the Laws of Logs to expand each expression:**

A. $\ln(ax^3) = \ln a + \ln x^3 = \ln a + 3\ln x$

B. $\log \frac{xy}{z} = \log(xy) - \log z = \log x + \log y - \log z$

C. $\ln \frac{a}{bc} = \ln a - \ln(bc) = \ln a - [\ln b + \ln c] = \ln a - \ln b - \ln c$

D. $\log_2 \frac{cd^3}{e^4} = \log_2 cd^3 - \log_2 e^4 = \log_2 c + 3\log_2 d - 4\log_2 e$

E. $\ln\left(\frac{\sqrt{x}\ \sqrt[3]{y}}{e^z\ \sqrt[5]{w^3}}\right) = \ln\sqrt{x}\ \sqrt[3]{y} - \ln e^z\ \sqrt[5]{w^3}$

$= \ln\sqrt{x} + \ln\sqrt[3]{y} - \left[\ln e^z + \ln\sqrt[5]{w^3}\right]$

$= \frac{1}{2}\ln x + \frac{1}{3}\ln y - z - \frac{3}{5}\ln w$

<u>EXAMPLE 5:</u> **Use the Laws of Logs to condense each expression into a single log with coefficient 1:**

A. $\log x + \log y = \log(xy)$

B. $\ln a - \ln b = \ln\frac{a}{b}$

C. $3\log_2 T = \log_2 T^3$

D. $\frac{3}{7}\log n = \log n^{3/7} = \log\sqrt[7]{n^3}$

E. $\frac{1}{2}\ln x + 5\ln y = \ln\sqrt{x} + \ln y^5 = \ln\left(\sqrt{x}\,y^5\right)$

F. $2\log x + \frac{1}{2}\log y + \frac{2}{5}\log z = \log x^2 + \log\sqrt{y} + \log\sqrt[5]{z^2}$

$= \log\left(x^2\sqrt{y}\ \sqrt[5]{z^2}\right)$

G. $\ln x + \ln y - \ln z = \ln\frac{xy}{z}$

H. $2\log x - \frac{1}{3}\log y = \log x^2 - \log y^{1/3} = \log\frac{x^2}{\sqrt[3]{y}}$

I. $\ln a - \ln b - \ln c = \ln\frac{a}{b} - \ln c = \ln\frac{\frac{a}{b}}{c} = \ln\frac{a}{bc}$

Alternate approach:

$\ln a - \ln b - \ln c = \ln a - (\ln b + \ln c) = \ln a - \ln(bc) = \ln\frac{a}{bc}$

J. $\frac{2}{3}\log(a+b)-\frac{1}{2}\log(a-b)-7\log(ab)$

$$= \log\sqrt[3]{(a+b)^2}-\log\sqrt{a-b}-\log(ab)^7$$

$$= \log\frac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}}-\log a^7b^7$$

$$= \log\frac{\frac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}}}{a^7b^7}$$

$$= \log\frac{\sqrt[3]{(a+b)^2}}{a^7b^7\sqrt{a-b}}$$

Homework

7. Expand each expression:

a. $\ln(ab)$ b. $\log(wyz)$ c. $\log_2(abcd)$

d. $\log\frac{h}{k}$ e. $\log\frac{a+b}{c}$ f. $\log_8\frac{x-y}{u+v}$

g. $\ln x^3$ h. $\log 7^n$ i. $\log_5 5^7$

8. Expand each expression:

a. $\log_7\sqrt{x}$ b. $\log\sqrt[3]{x}$ c. $\ln\sqrt[5]{Q^2}$

d. $\log\frac{1}{x^3}$ e. $\ln\frac{1}{u^{3/2}}$ f. $\log\frac{1}{T^{-4/5}}$

9. Expand each expression:

a. $\ln(a^2b)$
b. $\log(xy^3)$
c. $\ln(a^2b^3c^4)$
d. $\log_6 \frac{x^2}{y^3}$
e. $\ln \frac{ab}{c^9}$
f. $\log \frac{x^2y^5}{z^{10}}$

10. Expand each expression:

a. $\log_3 \frac{a}{bc}$
b. $\ln \frac{x^2y}{wz^3}$
c. $\log \frac{\sqrt{x}}{yz}$
d. $\log(a^2\, b \sqrt[7]{c})$
e. $\ln\left[\frac{wx}{yz}\right]^5$
f. $\log_5 \frac{a^2\sqrt[3]{y}}{wx}$

11. Condense each expression:

a. $\ln x - \ln 7$
b. $\log y + \log 12$
c. $\frac{1}{3}\ln 4 + \ln 2$
d. $\frac{2}{5}\log t - \frac{1}{5}\log t$
e. $\ln(x^2) + \ln x + \ln 7$
f. $\log x - \log y + \log z$
g. $\ln a - \ln b - 3\ln c$
h. $2\log a - \frac{1}{2}\log b - \frac{2}{3}\log c$

12. Evaluate each expression <u>without a calculator</u>:

a. $\log 10^{100}$
b. $e^{\ln 7}$
c. $\log 5 + \log 2$
d. $\ln \frac{e^{10}}{e^2}$
e. $\log_{24} 8 + \log_{24} 3$
f. $\ln x + \ln \frac{1}{x}$
g. $\log 50 + \log 20$
h. $\log_3 \frac{1}{2} + \log_3 162$
i. $\log_5 50 - \log_5 2$

13. True or False:

a. $\log 10^9 \;=\; 9$
b. $7^{\log_7 R} \;=\; 7^R$
c. $\ln e^{abc} \;=\; \ln(abc)$
d. $\ln(xy) \;=\; \ln x + \ln y$
e. $\log a^b \;=\; (\log a)^b$
f. $\ln \frac{a}{b} \;=\; \frac{\ln a}{\ln b}$
g. $\ln(x+y) \;=\; \ln x + \ln y$
h. $\log(a-b) \;=\; \log a - \log b$
i. $\ln \frac{1}{x} \;=\; -\ln x$
j. $\log_{12}(x^ab) \;=\; (a\log_{12} x)(\log_{12} b)$

k. $\ln(xy^z) = z\ln(xy)$

l. $\log_3 x - \log_3 y = \log_3 \frac{x}{y}$

m. $\log a + \log b = \log(ab)$

n. $\ln e^t = t$

o. $10^{\log 9} = 9$

p. $\ln \frac{1}{x} = \frac{1}{\ln x}$

q. $e^{\ln(e-3)} = e-3$

r. $\ln \frac{x}{y} = \ln x - \ln y$

s. $\log(ab^n) = \log a + n\log b$

t. $\ln \frac{a^2}{b^5} = \frac{2\ln a}{5\ln b}$

❑ *THE RICHTER SCALE FOR EARTHQUAKES*

In 1935, Charles Richter devised a scale for earthquakes called the Richter scale (what a coincidence!). If E represents the energy (in joules) of the earthquake, then the Richter magnitude M is given by

$$\boxed{M = \frac{2}{3}\log\left(\frac{E}{2.5\times 10^4}\right)}$$

EXAMPLE 6: **The energy release of the 1906 San Francisco earthquake was 5.96 × 10^{16} joules. Find the Richter magnitude of the quake.**

Solution: According to Richter's formula,

$$\begin{aligned} M &= \frac{2}{3}\log\left(\frac{5.96\times 10^{16}}{2.5\times 10^4}\right) \\ &= \frac{2}{3}\log\left(2.384\times 10^{12}\right) \\ &= \frac{2}{3}(12.3773) \\ &= \boxed{8.25} \end{aligned}$$

Homework

14. An earthquake releases 1.75×10^{11} joules of energy. What is the Richter magnitude of the quake?

15. A more serious quake releases 100 times as much energy as the one in the previous problem. Find the Richter magnitude.

☐ PROOFS OF THE THREE LAWS OF LOGS

The laws of logs we've developed and used in this chapter were arrived at using a few examples and noting a consistent pattern. We know that all the examples in the world never constitute a genuine proof, so now it's time to set the record straight and prove the laws the right way. There's no homework for this section, so ask your instructor what you're responsible for on the test.

THEOREM: **The First Law of Logs**

$$\log_b(xy) = \log_b x + \log_b y$$

PROOF: Let $w = \log_b x$ and $z = \log_b y$. Then

$x = b^w$ and $y = b^z$ (from the definition of log).

To prove the First Law of Logs, we'll begin with the left-hand side of the equation, substitute the results just obtained, and work our way to the right-hand side of the equation.

$$\begin{array}{lll} & \log_b(xy) & \text{(the left side of the formula)} \\ = & \log_b(b^w \cdot b^z) & \text{(substituting from the results above)} \\ = & \log_b(b^{w+z}) & \text{(the first law of exponents)} \end{array}$$

$$= \quad w + z \qquad \text{(the first log cancelation rule)}$$

$$= \quad \log_b x + \log_b y \qquad \text{(substituting back)}$$

Q.E.D.

"The log of a <u>product</u> is the <u>sum</u> of the logs."

THEOREM: **The Second Law of Logs**

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

PROOF: Let $w = \log_b x$ and $z = \log_b y$. Then $x = b^w$ and $y = b^z$. We proceed like the previous proof:

$$\log_b\left(\frac{x}{y}\right) \qquad \text{(the left side of the formula)}$$

$$= \quad \log_b\left(\frac{b^w}{b^z}\right) \qquad \text{(substituting from the results above)}$$

$$= \quad \log_b(b^{w-z}) \qquad \text{(a law of exponents)}$$

$$= \quad w - z \qquad \text{(the first log canceling rule)}$$

$$= \quad \log_b x - \log_b y \qquad \text{(substituting back)}$$

Q.E.D.

"The log of a <u>quotient</u> is the <u>difference</u> of the logs."

The Third Law of Logs is used whenever there's an exponent on the quantity we're taking the log of.

THEOREM: **The Third Law of Logs**

$$\log_b x^n = n\log_b x$$

PROOF: As in the two previous proofs, let $w = \log_b x$, which implies that $x = b^w$. Thus,

$\log_b x^n$ (the left side of the formula)

$= \log_b (b^w)^n$ (substituting the expression for x)

$= \log_b (b^{wn})$ (one of the laws of exponents)

$= wn$ (the first log canceling rule)

$= nw$ (commutative property for multiplication)

$= n\log_b x$ (substituting back) **Q.E.D.**

"To calculate the log of a <u>power</u>, bring down the power as a <u>coefficient</u>."

Review Problems

16. $\ln e + \log 1 - \log_3 3 - \log_7 49 \ =$

17. a. $\log_{12} 12^N \ =$ b. $9^{\log_9 56} \ =$
c. $\ln e^{a-b} \ =$ d. $\log 10^{25} \ =$
e. $\log_3 5^y \ =$ f. $5^{\log_5(-5)} \ =$

18. $\log_b b^x \quad b^{\log_b x} \ =$

19. $\ln(\log 10) \ =$

20. Expand: $\log\left(\dfrac{ab}{10}\right)$

$$y = \log_b x \text{ means } b^y = x$$

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^n = n\log_b x$$

21. Expand: $\ln\left(\frac{x^3}{y\sqrt{z}}\right)$

22. Condense: $\ln x + \ln y - \ln z$

23. Condense: $3\log x + \frac{2}{3}\log y - \frac{1}{2}\log z$

24. True/False:

a. $\ln(a-b) = \ln a - \ln b$

b. $\log e^t = t$

c. $\ln\left(\frac{2}{x}\right) = \frac{\ln 2}{\ln x}$

d. $\ln(ab^{10}) = 10\ln(ab)$

25. The energy release of an earthquake is 1.23×10^{17} joules. Use the formula $M = \frac{2}{3}\log\left(\frac{E}{2.5\times 10^4}\right)$ to find the Richter magnitude of the quake.

26. True/False:

a. $\log 10^{a+b} = a+b$

b. $\log e^{u-w} = u-w$

c. $6^{\log_6 z} = z$

d. $e^{\log Q} = Q$

e. $\ln(ab) = \ln a + \ln b$

f. $\ln\frac{x}{y} = \frac{\ln x}{\ln y}$

g. $\log ab^y = y\log ab$

h. $\log ab^y = \log a + y\log b$

i. $\ln p - \ln q = \ln\frac{p}{q}$

j. $\log_3 32 - \log_3 8 = \log_3 4$

k. $\ln(x+y+z) = \ln x + \ln y + \ln z$

l. $\log 7^7 = 7\log 7$

m. $(\ln x)^3 = 3\ln x$

n. $\frac{2}{3}\log x + 4\log y = \log\left(\sqrt[3]{x^2}\, y^4\right)$

o. If the energy release of an earthquake is 2.807×10^{15} joules, then the Richter number of the earthquake is about 7.37.

☐ *To ∞ and Beyond*

Prove that $\ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) = 4\ln(x^2+2)$

Solutions

1. Because the first operation in $b^{\log_b x}$ is $\log_b x$, whose domain is $(0, \infty)$.

2. a. 1 b. 1 c. $u + v$ d. xyz e. 500 f. xy g. log 7 h. 1
i. $a - b$ j. R k. As is l. As is

3. $\log_2(8+8) = \log_2 16 = \mathbf{4}$, whereas
$\log_2 8 + \log_2 8 = 3+3 = \mathbf{6}$

4. $\log(xy) = \log(100 \cdot 1{,}000) = \log 100{,}000 = \mathbf{5}$, but
$(\log x)(\log y) = (\log 100)(\log 1{,}000) = 2 \cdot 3 = \mathbf{6}$.

5. a. log 10,000 b. log 1,000,000 c. log 10
d. 1,000,000 e. $\log_2 512$ f. $\log_3 27$

g. $\log_4 256$ h. $\log 1{,}000{,}000$ i. $\ln e^{12}$
j. $\ln e^{20}$

6. a. $\log 100$ b. $\log 100$ c. $\log_5 5$
d. $\ln e^3$ e. $\log_6 6$ f. $\ln e^{15}$
g. $\log 10$ h. $\log_8 8$ i. $\log_2 4$
j. $\log_{12} 144$

7. a. $\ln a + \ln b$ b. $\log w + \log y + \log z$
c. $\log_2 a + \log_2 b + \log_2 c + \log_2 d$ d. $\log h - \log k$
e. $\log(a+b) - \log c$ f. $\log_8(x-y) - \log_8(u+v)$
g. $3\ln x$ h. $n\log 7$ i. 7

8. a. $\frac{1}{2}\log_7 x$ b. $\frac{1}{3}\log x$ c. $\frac{2}{5}\ln Q$
d. $-3\log x$ e. $-\frac{3}{2}\ln u$ f. $\frac{4}{5}\log T$

9. a. $2\ln a + \ln b$ b. $\log x + 3\log y$
c. $2\ln a + 3\ln b + 4\ln c$ d. $2\log_6 x - 3\log_6 y$
e. $\ln a + \ln b - 9\ln c$ f. $2\log x + 5\log y - 10\log z$

10. a. $\log_3 a - \log_3 b - \log_3 c$ b. $2\ln x + \ln y - \ln w - 3\ln z$
c. $\frac{1}{2}\log x - \log y - \log z$ d. $2\log a + \log b + \frac{1}{7}\log c$
e. $5\ln w + 5\ln x - 5\ln y - 5\ln z$ f. $2\log_5 a + \frac{1}{3}\log_5 y - \log_5 w - \log_5 x$

11. a. $\ln\frac{x}{7}$ b. $\log(12y)$ c. $\ln(2\sqrt[3]{4})$ d. $\log\sqrt[5]{t}$
e. $\ln(7x^3)$ f. $\log\frac{xz}{y}$ g. $\ln\frac{a}{bc^3}$ h. $\log\frac{a^2}{\sqrt{b}\,\sqrt[3]{c^2}}$

12. a. 100 b. 7 c. 1 d. 8 e. 1
f. 0 g. 3 h. 4 i. 2

13. a. T b. F c. F d. T e. F f. F g. F h. F i. T
j. F k. F l. T m. T n. T o. T p. F q. F (tricky!)
r. T s. T t. F

14. $M = 4.6$

15. $M = 5.9$. Notice that this magnitude is only 1.3 Richter points higher than the previous answer, yet the earthquake was 100 times more powerful. Even a small difference in the Richter magnitude represents a huge difference in the actual power of the earthquake.

16. -2

17. a. N b. 56 c. $a - b$ d. 25 e. As is f. Undefined

18. x^2 **19**. 0 **20**. $\log a + \log b - 1$

21. $3\ln x - \ln y - \frac{1}{2}\ln z$ **22**. $\ln\left(\frac{xy}{z}\right)$

23. $\log\left[\frac{x^3\sqrt[3]{y^2}}{\sqrt{z}}\right]$ **24**. They're all false. **25**. 8.46

26. a. T b. F c. T d. F e. T f. F g. F h. T
i. T j. T k. F l. T m. F n. T o. T

"Nothing can stop the man with the right mental attitude from achieving his goal; nothing on earth can help the man with the wrong mental attitude."

Thomas Jefferson

CH NN – LOGARITHMS

❑ *MELANIE'S ALLOWANCE*

Melanie tells her father that she will pay the entire cost (room and board) of her entire college education *if* he agrees to the following plan: He gives her 2¢ on the first day of the month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on till the 30th of the month. After that month – no more money. Dad (who was a philosophy major) thinks this is a great money-saving idea and accepts Melanie's proposal.

Day	# Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
⋮	⋮
30	1,073,741,824 (almost $11 million)

In the chart, we have calculated Melanie's earnings for each of the first 10 days of the month. Take your calculator and verify the number of pennies Melanie will receive on the 30th day.

Now let's come up with a direct formula that computes the money earned from the day of the month, <u>without knowing all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day. For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$.

Now, for the 30th day, calculate 2^{30}, and you should get **1,073,741,824** (which, remember, is the number of <u>pennies</u> Melanie gets on day 30;

you can divide this number of pennies by 100, and then round off to see that it’s equivalent to about 10.7 million dollars!

Homework

1. a. How many pennies will Melanie earn on the 15th day of the month?

   b. How many pennies will Melanie earn on the 31st of the month if Dad agrees to extend the plan that far?

   c. On which day of the month did Melanie earn <u>half</u> of what she earned on the 30th day?

2. Now for a little practice on the concept to be covered in this chapter. I’ll give you a penny amount, and you tell me which day of the month Melanie earned that amount of money.

   a. 512¢ b. 4,096¢ c. 1,048,576¢ d. 33,554,432¢

3. Similar question, but a little more abstract: I’ll give you a number, and you tell me the *exponent* that **<u>2</u>** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 2,048, then you say “11” because $2^{\boxed{11}} = 2{,}048$.

   a. 2 b. 256 c. 64 d. 1
   e. 8,192 f. 131,072 g. 524,288 h. 1/2

4. Another question like #3, but now when I give you the number, you tell me the *exponent* that **<u>10</u>** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 1,000, then you say “3” because $10^{\boxed{3}} = 1{,}000$.

   a. 100 b. 10,000 c. 10 d. 1

   e. 100,000 f. 1,000,000 g. 1 billion h. 1 googol

❑ THE MEANING OF A LOG

A ***log*** (short for ***logarithm***) is an exponent. It's the exponent to which one number (called the base) must be raised to get a specified number. This definition is so far off in the clouds that we need an example right now.

logarithm

from the Greek:

logos = logic

arithmos = number

For our first example, to calculate

$\log_{10}(1000)$ [read: "log, base 10, of 1000" or "log of 1000, base 10]

we ask ourselves, "10 raised to *what power* equals 1000?" In other words, 10 to the "what" equals 1000? The answer is 3, since $10^3 = 1000$. Therefore,

$\log_{10}(1000) = \mathbf{3}$ [log, base 10, of 1000 is 3.]

For a second example, let's analyze $\log_2 32$, which asks us, "2 raised to the "what" equals 32?" Well, 2 to the 5th power equals 32, and so

$\log_2(32) = \mathbf{5}$ [log, base 2, of 32 is 5.]

Our third example will describe a log a little differently: If you can fill in the box in the equation

$$4^{\square} = 16$$

then you have found the "log, base 4, of 16," which is 2. That is,

$$\log_4(16) = \mathbf{2}$$

Notation: A log is a function, so notation like $\log_2(128)$ certainly makes sense, just like when we use parentheses in function notation:

$f(x)$. But if it's clear what we're taking the log of, we don't really need the parentheses; so, for example, $\log_2(128)$ is simply written $\log_2 128$. [Although in computer programming, the parentheses are required.]

Summary: $\log_{10} 1000 = \mathbf{3}$ because $10^{\mathbf{3}} = 1000$

$\log_2 32 = \mathbf{5}$ because $2^{\mathbf{5}} = 32$

$\log_4 16 = \mathbf{2}$ because $4^{\mathbf{2}} = 16$

This is really abstract, isn't it? Let's get right to some homework.

Homework

5. To find $\log_5 25$, which is read "log, base 5, of 25," ask yourself, "5 raised to what power equals 25?" $5^{\square} = 25$

6. To find $\log_2 8$, which is read "log, base 2, of 8," ask yourself, "2 raised to what power equals 8?" $2^{\square} = 8$

7. To find $\log_9 9$, which is read "log, base 9, of 9," ask yourself, "9 raised to what power equals 9?" $9^{\square} = 9$

8. To find $\log_{17} 1$, which is read "log, base 17, of 1," ask yourself, "17 raised to what power equals 1?" $17^{\square} = 1$

9. To find $\log_{100} 10$, which is read "log, base 100, of 10," ask yourself, "100 raised to what power equals 10?" $100^{\square} = 10$

10. To find $\log_6\left(\frac{1}{6}\right)$, which is read “log, base 6, of 1/6,” ask yourself, “6 raised to what power equals $\frac{1}{6}$?” $6^{\square} = \frac{1}{6}$

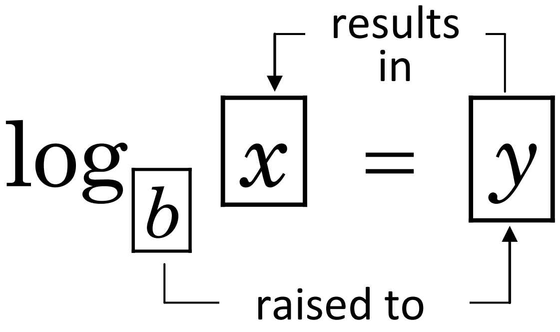
❑ *THE OFFICIAL DEFINITION OF LOGARITHM*

$$\boxed{\log_b x = y \quad \text{means} \quad b^y = x}$$

The notation “$\log_b x$” is read either

“log, base *b,* of *x*” or “log of *x*, base *b*”

Here’s another way to visualize the meaning of logarithm:



EXAMPLE 1:

A. $\log_{10} 10{,}000 = \mathbf{4}$ Why? Because $10^4 = 10{,}000$

B. $\log_e e^2 = \mathbf{2}$ because $e^2 = e^2$

C. $\log_{17} 17 = \mathbf{1}$ because $17^1 = 17$

D. $\log_e 1 = \mathbf{0}$ because $e^0 = 1$

E. $\log_{25} 5 = \mathbf{\frac{1}{2}}$ because $25^{1/2} = \sqrt{25} = 5$

F. $\log_{64} 4 = \mathbf{\frac{1}{3}}$ because $64^{1/3} = \sqrt[3]{64} = 4$

G. $\log_8 4 = \mathbf{\frac{2}{3}}$ because $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

H. $\log_{13}\left(\frac{1}{13}\right) = -\mathbf{1}$ because $13^{-1} = \frac{1}{13}$

I. $\log_6\left(\frac{1}{36}\right) = -\mathbf{2}$ because $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

J. $\log_{49}\left(\frac{1}{7}\right) = -\mathbf{\frac{1}{2}}$ because $49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

K. $\log_{1/2}\left(\frac{1}{8}\right) = \mathbf{3}$ because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

<u>EXAMPLE 2:</u>

A. $\log_b b = \mathbf{1}$ since $b^1 = b$

B. $\log_b 1 = \mathbf{0}$ since $b^0 = 1$

C. $\log_b \frac{1}{b} = -1$ since $b^{-1} = \frac{1}{b}$

D. $\log_b \sqrt{b} = \frac{1}{2}$ since $b^{1/2} = \sqrt{b}$

E. $\log_b \sqrt[n]{b} = \dfrac{1}{n}$ since $b^{1/n} = \sqrt[n]{b}$

E. $\log_b b^n = \boldsymbol{n}$ since $b^n = b^n$

Homework

Find the value of each log:

11. a. $\log_{10} 100$ b. $\log_5 125$ c. $\log_8 64$ d. $\log_2 64$

12. a. $\log_e e^5$ b. $\log_b b^2$ c. $\log_{\sqrt{2}} \sqrt{2}$ d. $\log_L L$

13. a. $\log_{10} 1$ b. $\log_e 1$ c. $\log_{\sqrt[5]{99}} 1$ d. $\log_b 1$

14. a. $\log_{36} 6$ b. $\log_{49} 7$ c. $\log_{144} 12$ d. $\log_b \sqrt{b}$

15. a. $\log_5\left(\frac{1}{5}\right)$ b. $\log_e\left(\frac{1}{e}\right)$ c. $\log_{1/e} 1$ d. $\log_n\left(\frac{1}{n}\right)$

16. a. $\log_Q Q^n$ b. $\log_x 1$ c. $\log_{2.3} 2.3$ d. $\log_9 81$

17. a. $\log_8 2$ b. $\log_{64} 4$ c. $\log_{125} 5$ d. $\log_a \sqrt[3]{a}$

❑ *Calculating Logs*

The homework problems above were designed to be solved by inspection (with a little experimentation and insight). Some logs aren't easy to do that way. So now we present a longer, but more systematic way, of evaluating a log by solving a certain exponential equation.

<u>EXAMPLE 3:</u> **Calculate: $\mathbf{\log_{27} 9}$**

<u>Solution:</u> Let's give our log expression a name — call it y. Now we can write an equation:

$$\log_{27} 9 = y$$

The definition of log shows us how we can convert our log equation into an exponential equation:

$$27^y = 9$$

And now we solve for y. The chapter *Exponential Equations (No Logs)* showed us how:

$$27^y = 9 \Rightarrow (3^3)^y = 3^2 \Rightarrow 3^{3y} = 3^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

But y was the name we gave to the original log problem. So we can conclude that

$$\boxed{\log_{27} 9 = \frac{2}{3}}$$

To **check** our result, we can raise 27 to the $\frac{2}{3}$ power and make sure it comes out 9:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9 \quad ✓$$

Homework

Find the value of each log:

18. $\log_{64} 16$
19. $\log_{25}\left(\frac{1}{5}\right)$
20. $\log_{16} 8$
21. $\log_{27}\left(\frac{1}{9}\right)$
22. $\log_8 16$
23. $\log_{32}\left(\frac{1}{2}\right)$
24. $\log_{1/3}\left(\frac{1}{9}\right)$
25. $\log_{1/4} 64$
26. $\log_9 1$
27. $\log_\pi \pi^5$
28. $\log_\pi \sqrt{\pi}$
29. $\log_4\left(\frac{1}{8}\right)$

30. $\log_5\left(\frac{1}{25}\right)$

31. $\log_3\left(\frac{1}{\sqrt{3}}\right)$

32. $\log_4\left(\frac{1}{64}\right)$

33. $\log_2\left(\frac{1}{128}\right)$

34. $\log_{10}\sqrt{10}$

35. $\log_{10}\sqrt[3]{100}$

36. $\log_{10}\left(\frac{1}{100}\right)$

37. $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$

38. $\log_{10}\left(\frac{1}{\sqrt{1000}}\right)$

☐ THE pH SCALE FOR ACIDS AND BASES

One of the uses of logs is defining the **pH scale** for acids and bases. The official definition of the pH of a substance is the negative **logarithm** (base 10) of the hydrogen ion concentration of the substance. Acids (like lemonade) have a pH smaller than 7, while bases (like Drano) have a pH higher than 7. The pH of pure water is a neutral 7. [The word *alkali* is another term for base.]

We'll use the official chemistry notation for the hydrogen ion concentration, $[H^+]$, which has the units of moles/liter. It is not necessary to understand any of the chemistry — indeed, the math takes care of everything. Devised by a biochemist while working on the brewing of beer, the **pH** of a substance is defined to be the negative logarithm (base 10) of the hydrogen ion concentration:

$$\boxed{\text{pH} = -\log_{10}[\text{H}^+]}$$

Logs are also used in the definition of the *Richter Scale* for earthquakes, and for the *decibel scale* for measuring the loudness of sound.

New Notation and Calculator Hints:

1. On a TI-30, to enter a number like 1.6×10^{-13}, first press 1.6, then press the "EE" button, and then press 13, and last the +/– key. Your display should then look something like $1.6\ ^{-13}$ (the base of 10 is understood).

2. To find $\log_{10} 1000$, enter 1000 into your calculator and then press the ***log*** button. You should, of course, get an answer of 3. On newer calculators, try pressing the *log* button first, followed by 1000. Using 10 as a base for logs is so "common" that it is officially referred to as the **common log**, and we dispense with writing the base, 10 — it's "understood":

$$\log_{10} x \text{ is written } \log x$$

3. [Skip the following if you've never come across the number ***e***.] A base of ***e*** is so important and occurs so "naturally" in the universe that "log, base ***e***" also gets its own name and notation:

$$\log_e x \text{ is written } \ln x$$

You read "ln *x*" either as "el en *x*" or "el en of *x*" or "the **natural log** of *x*." Teachers will many times write it on the whiteboard in cursive:

ln x (*l* for **l**og, *n* for **n**atural)

EXAMPLE 4: **A sample of orange juice has a hydrogen ion concentration of 2.9×10^{-4} moles/liter. Find the pH of the orange juice.**

Solution: According to the definition,

$$\text{pH} = -\log_{10}[\text{H}^+] = -\log(2.9\times10^{-4}) \approx -(-3.54) = 3.54$$

According to the text next to the lemonade stand above, this should mean that orange juice is an acid, as indeed it is (the sour taste is one clue). Thus, the pH of the orange juice is

$$\boxed{3.54}$$

Homework

39. The hydrogen ion concentration of household ammonia is 1.26×10^{-12} moles/liter. Find the pH of the ammonia. Is it an acid or a base?

40. Pure water has a hydrogen ion concentration of 1.0×10^{-7} moles/liter. Prove that water has a neutral pH of 7.

41. Find the pH of each substance given its molarity:

a. 1.3×10^{-2} moles/L
b. 2.8×10^{-6} moles/L
c. 0.3×10^{-10} moles/L
d. 9.2×10^{-12} moles/L
e. 5.9×10^{-7} moles/L
f. 8.0×10^{-1} moles/L

Review Problems

42.
a. $\log_{10} 1{,}000{,}000 =$
b. $\log_3 3^7 =$
c. $\log_e e =$
d. $\log_7 1 =$
e. $\log_{16} 4 =$
f. $\log_9\left(\frac{1}{9}\right) =$
g. $\log_5\left(\frac{1}{25}\right) =$
h. $\log_{36}\left(\frac{1}{6}\right) =$
i. $\log_2 512 =$

43. a. $\log_a a =$ b. $\log_c 1 =$ c. $\log_7 7^N =$

44. The hydrogen ion concentration of an unknown liquid is 3.4×10^{-11} moles/L. Find the pH of the liquid.

45. What kind of music do they play in a lumber camp?

Solutions

---

1. a. 32,768 b. 2,147,483,648 c. I'm not gonna tell you that.

2. a. day 9 b. day 12 c. day 20 d. day 25

3. a. 1 b. 8 c. 6 d. 0 e. 13 f. 17 g. 19 h. –1

4. a. 2 b. 4 c. 1 d. 0 e. 5 f. 6 g. 9 h. 100

5. $5^{\boxed{?}} = 25$; since $5^2 = 25, \log_5 25 = 2$.

6. $2^{\boxed{?}} = 8$; since $2^3 = 8, \log_2 8 = 3$.

7. 1 **8**. 0

9. $100^{\boxed{?}} = 10$; since $100^{1/2} = \sqrt{100} = 10, \log_{100} 10 = \frac{1}{2}$.

10. –1

11. a. 2 b. 3 c. 2 d. 6 **12**. a. 5 b. 2 c. 1 d. 1

13. a. 0 b. 0 c. 0 d. 0 **14**. a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{2}$

15. a. -1 b. -1 c. 0 d. -1 **16**. a. n b. 0 c. 1 d. 2

17. a. $\frac{1}{3}$ b. $\frac{1}{3}$ c. $\frac{1}{3}$ d. $\frac{1}{3}$

18. Let $y = \log_{64} 16 \;\Rightarrow\; 64^y = 16 \;\Rightarrow\; (4^3)^y = 4^2$

$\Rightarrow\; 4^{3y} = 4^2 \;\Rightarrow\; 3y = 2 \;\Rightarrow\; y = \frac{2}{3}$

19. $-\frac{1}{2}$ **20**. $\frac{3}{4}$ **21**. $-\frac{2}{3}$ **22**. $\frac{4}{3}$ **23**. $-\frac{1}{5}$ **24**. 2

25. -3 **26**. 0 **27**. 5 **28**. $\frac{1}{2}$ **29**. $-\frac{3}{2}$

30. -2 **31**. $-\frac{1}{2}$ **32**. -3 **33**. -7 **34**. $\frac{1}{2}$

35. $\frac{2}{3}$ **36**. -2 **37**. $-\frac{1}{2}$ **38**. $-\frac{3}{2}$

39. pH = 11.9; it's a base (an alkali), since its pH is greater than 7.

40. $\text{pH} = -\log[\text{H}^+] = -\log(1.0\times10^{-7}) = -(-7) = 7$

41. a. 1.89 b. 5.55 c. 10.52 d. 11.04 e. 6.23 f. 0.10

42. a. 6 b. 7 c. 1 d. 0 e. $\frac{1}{2}$ f. -1 g. -2 h. $-\frac{1}{2}$ i. 9

43. a. 1 b. 0 c. N

44. 10.47 (an alkali, or base)

45. I just can't give the answer away.

"Learning is not the product of teaching.

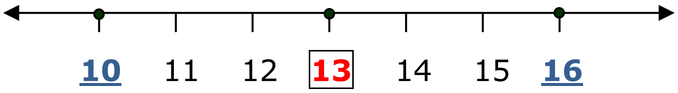
Learning is the product of the activity of learners."

– John Holt, American psychologist and educator

CH NN –MIDPOINT ON THE LINE AND IN THE PLANE

❑ MIDPOINT ON THE LINE

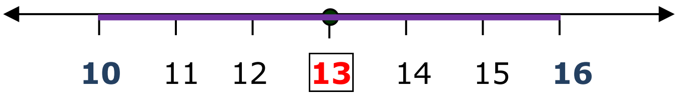
Here's a question for you: What number is *midway* between 10 and 16? You probably know that the number is 13. Why? Because 13 is 3 units away from 10, and 13 is also 3 units away from 16.



Now we need a simple way to find the number that is midway between any two numbers, even when the numbers are not nice, or worse yet, when there are variables involved. Notice this: If we take the **average** (officially called the *arithmetic mean*) of 10 and 16 — by adding the two numbers and dividing by 2 — we get

$$\frac{10+16}{2} = \frac{26}{2} = \mathbf{13}\text{, the midway number}$$

Let's rephrase what we've done with some new terminology. Consider the *line segment* connecting 10 and 16 on the number line:



We can now refer to the 13 as the ***midpoint*** of the line segment connecting 10 and 16.

What is the *midpoint* of the line segment connecting –2.8 and 14.6? Just calculate the average of –2.8 and 14.6:

$$\frac{-2.8+14.6}{2} = \frac{11.8}{2} = \mathbf{5.9}$$

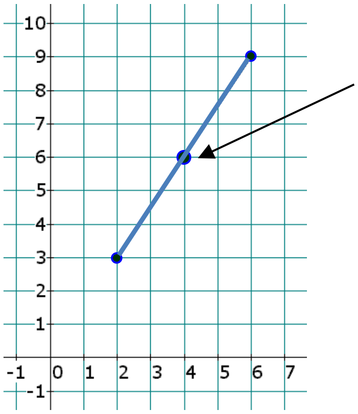
When you see the term ***midpoint***, think *average*!

Homework

1. Find the **midpoint** of the line segment connecting the two given numbers on a number line:

a. 10 and 20
b. 13 and 22
c. –8 and –26
d. –3 and 7
e. –7 and 6
f. π and –π
g. –21 and –99
h. 0 and 43
i. –50 and 0
j. –44 and 19
k. –41 and 88
l. $3x$ and $-x$

☐ *MIDPOINT IN THE PLANE*



The **midpoint** is found by averaging the *x*-coordinates and then averaging the *y*-coordinates.

Now for the more important question: Consider the two points (2, 3) and (6, 9) in the plane and the line segment that connects them. We need to figure out what point is the *midpoint* of the line segment connecting the two points. Recall the advice given above: When you see midpoint, think *average*. So the x-coordinate of the midpoint is the average of the x-coordinates of the two endpoints:

$$x = \frac{2+6}{2} = \frac{8}{2} = \mathbf{4}$$

And the y-coordinate of the midpoint is the average of the y-coordinates of the two endpoints:

$$y = \frac{3+9}{2} = \frac{12}{2} = \mathbf{6}$$

We conclude that the midpoint is **(4, 6)**. That's all there is to it. Now let's do a complete example without plotting any points or drawing any segments.

EXAMPLE 1: **Find the midpoint of the line segment connecting the points (–42, –33) and (90, –10).**

Solution: No graphing needed — we have a formula. The x-coordinate of the midpoint is found by averaging the x-coordinates of the two given points:

$$x = \frac{-42+90}{2} = \frac{48}{2} = 24$$

The y-coordinate of the midpoint is found by averaging the y-coordinates of the two given points:

$$y = \frac{-33+(-10)}{2} = \frac{-43}{2} = -\frac{43}{2}$$

The midpoint is therefore the point

$$\boxed{\left(24, -\frac{43}{2}\right)} \quad \text{or, } (24, -21.5)$$

Homework

2. Find the **midpoint** of the line segment connecting the given pair of points:

a. (–2, 5) and (2, 7)

b. (0, 1) and (0, 6)

c. (–5, 8) and (–5, –8)

d. (–2, 7) and (5, –3)

e. (–9, 2) and (–13, –40)

f. (0, 0) and (–6, –9)

g. (5, 4) and (5, 4)

h. (14, 0) and (0, –9)

i. (8, 8) and (–19, –19)

j. (π, 0) and (–π, 0)

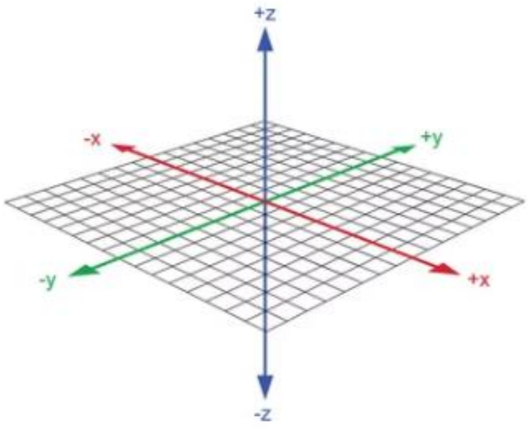
k. $(0, \sqrt{2})$ and $(0, -\sqrt{2})$

l. (a, b) and (c, d)

m. (a, b) and $(a, -b)$

n. $(3a, 3b)$ and $(-3a, b)$

☐ *TO ∞ AND BEYOND*



A point on a 1-dimensional line can be described by a single number, for example, 7.

A point in a 2-dimensional plane can be described by an ordered pair, for example, (2, –9).

A point in 3-dimensional space (which has an x-axis, a y-axis, and a ***z*-axis**), can be described by an ordered triple, for example (1, –5, 12).

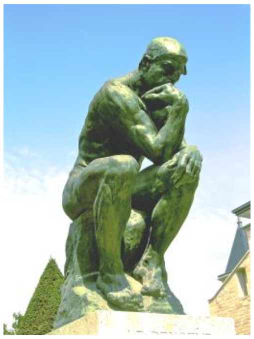
Find the **midpoint** of the line segment connecting the points (2, –3, 7) and (–5, 17, 20).

Solutions

1. a. 15 b. 17.5 c. –17 d. 2 e. –0.5 f. 0
g. –60 h. 21.5 i. –25 j. –25 k. 23.5 l. x

2. a. (0, 6) b. (0, 7/2) c. (–5, 0)
d. (3/2, 2) e. (–11, –19) f. (–3, –9/2)
g. (5, 4) h. (7, –9/2) i. (–11/2, –11/2)
j. (0, 0) k. (0, 0) l. $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$
m. (a, 0) n. (0, 2b)

> "What sculpture is to a block of marble, education is to the human soul."
>
> **Joseph Addison**



GRAPHING PARABOLAS

The **parabola** (accent on the *rab*) is a very special shape used in searchlights and satellite dishes. Even football sports reporters use parabolic reflectors to listen in on comments made by coaches on the sidelines and players in the huddle. In fact, when a football is thrown or punted, its path is that of a parabola.

☐ *GRAPHING A PARABOLA*

EXAMPLE 1: **Graph:** $y = x^2 - 4x + 3$

Solution: Who's to say that the graph of this formula isn't a straight line? Let's work it out; we'll find some points on our graph by choosing some values of x, and then calculate the corresponding y-values — and we'll see what points we get.

If $x = -1$, then $y = (\mathbf{-1})^2 - 4(\mathbf{-1}) + 3 = 1 + 4 + 3 = 8 \Rightarrow \mathbf{(-1, 8)}$

If $x = 0$, then $y = (\mathbf{0})^2 - 4(\mathbf{0}) + 3 = 0 - 0 + 3 = 3 \Rightarrow \mathbf{(0, 3)}$

If $x = 1$, then $y = (\mathbf{1})^2 - 4(\mathbf{1}) + 3 = 1 - 4 + 3 = 0 \Rightarrow \mathbf{(1, 0)}$

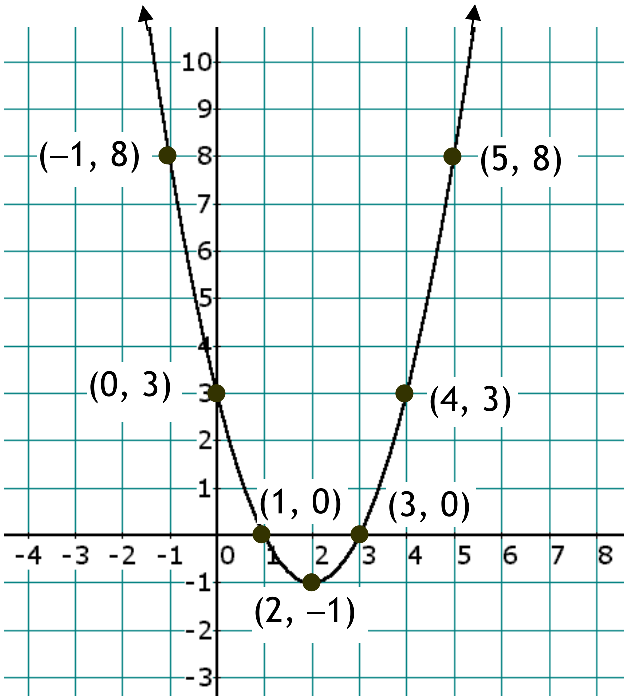
If $x = 2$, then $y = (\mathbf{2})^2 - 4(\mathbf{2}) + 3 = 4 - 8 + 3 = -1 \Rightarrow \mathbf{(2, -1)}$

If $x = 3$, then $y = (\mathbf{3})^2 - 4(\mathbf{3}) + 3 = 9 - 12 + 3 = 0 \Rightarrow \mathbf{(3, 0)}$

If $x = 4$, then $y = (\mathbf{4})^2 - 4(\mathbf{4}) + 3 = 16 - 16 + 3 = 3 \Rightarrow \mathbf{(4, 3)}$

If $x = 5$, then $y = (\mathbf{5})^2 - 4(\mathbf{5}) + 3 = 25 - 20 + 3 = 8 \Rightarrow \mathbf{(5, 8)}$

Plotting these seven points leads us to the following graph:



What do we notice about the graph? It's a curve, not a straight line. We notice that x can be any real number (but notice that y never goes below −1). Also note that the graph has one y-intercept but two x-intercepts. In addition, there is no highest point on the parabola, but the lowest point on the parabola is (2, −1), and we call this point the ***vertex*** of the parabola.

This is the shape called the ***parabola***. We say that the parabola just graphed "**opens *up***." The equation of a parabola is characterized by the fact that one variable (the x) is *squared* while the other variable (the y) is raised to the first power.

EXAMPLE 2: **Graph:** $y = -x^2 - 2x - 1$

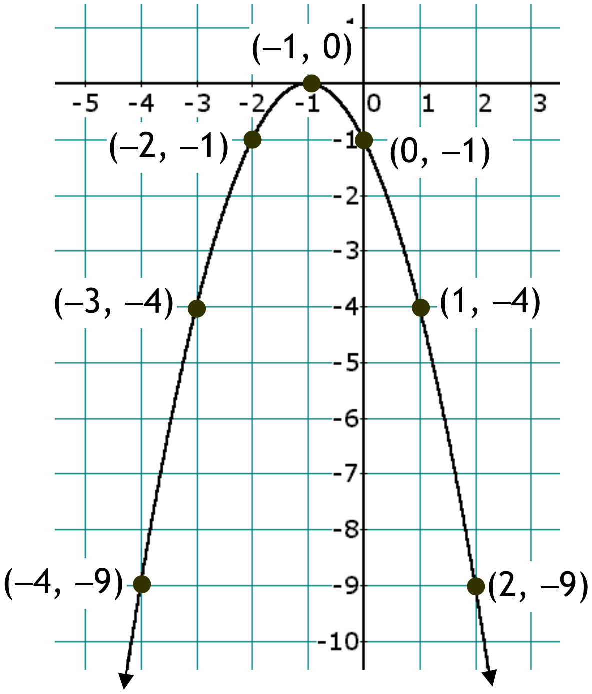
Solution: First we notice that the quadratic term, the $-x^2$, has a leading negative sign. And we remember that, due to the Order of Operations, exponents have a higher priority than minus signs, so we know that to evaluate $-x^2$, we square the x first, and apply the minus sign <u>second</u>. [For example, $-7^2 = -49$.] We'll calculate together two points on the parabola and leave the rest of the points for you to do.

If $x = 2$, then $y = -(\mathbf{2})^2 - 2(\mathbf{2}) - 1 = -4 - 4 - 1 = -9$. Thus, the point **(2, –9)** is on the graph of our parabola.

If $x = -3$, then $y = -(\mathbf{-3})^2 - 2(\mathbf{-3}) - 1 = -9 + 6 - 1 = -4$. Therefore, the point **(–3, –4)** is on the graph.

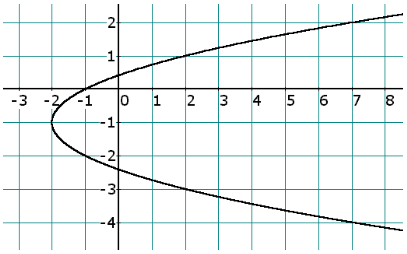
You should now do the calculations to show that each of the following points is also on the graph:

(–4, –9) **(–2, –1)** **(–1, 0)** **(0, –1)** **(1, –4)**



From the formula (and kind of from the graph) we note that *x* can be <u>any</u> real number, but that *y* never gets above 0. As for intercepts, there's one *x*-intercept and one *y*-intercept. There is no lowest point on the graph, but the highest point, the **vertex**, is the point (–1, 0). We say that this parabola **opens down**.

There are also "sideways" parabolas, but in this chapter only parabolas which open up or down will be discussed.



Homework

1. In Example 1 we saw that the graph of $y = x^2 - 4x + 3$ is a parabola opening up. Example 2 showed us that the graph of $y = -x^2 - 2x - 1$ is a parabola opening down. Take a guess what property of these equations determines whether the parabola opens up or down.

2. True/False: (Recall that all parabolas in this chapter open up or down.)

   a. Every parabola has a y-intercept.

   b. Every parabola has an x-intercept.

   c. Every parabola has a vertex.

   d. The vertex of a parabola is always the highest point of the parabola.

   e. The vertex of a parabola is always the lowest point of the parabola.

3. Graph each parabola by plotting points. Then use your graph to determine the intercepts and the vertex of your parabola:

   a. $y = 9 - x^2$

   b. $y = x^2 + 6x + 5$

   c. $y = x^2 + 2x - 8$

   d. $y = -x^2 + 4x - 4$

   e. $y = 0.5x^2$

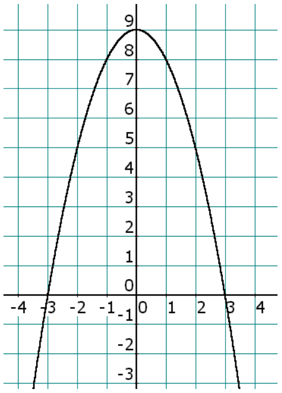
   f. $y = -0.2x^2 + 5$

Solutions

1. The coefficient of the quadratic term in Example 1 is positive, while that of the quadratic term in Example 2 is negative. That's the clue which determines whether a parabola opens up or down. Therefore, the parabola $y = \pi x^2 - 13x + 2$ opens up, whereas the parabola $y = -0.7x^2 + 99x + 14$ opens down.

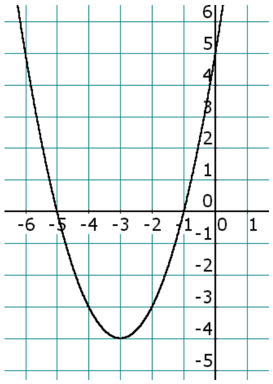
2. a. True b. False c. True d. False e. False

3. a.



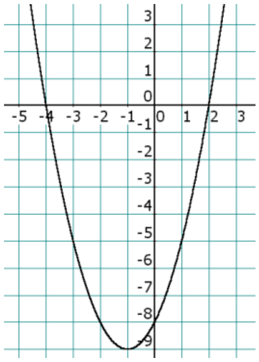
Intercepts:
(–3, 0), (3, 0), (0, 9)
Vertex: (0, 9)

b.



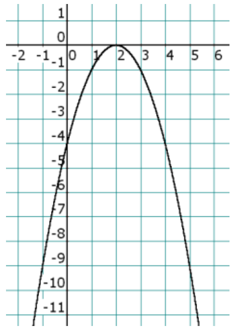
Intercepts:
(–5, 0), (–1, 0), (0, 5)
Vertex: (–3, –4)

c.



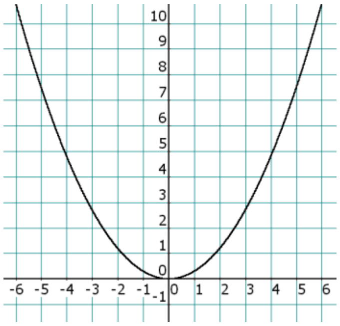
Intercepts:
(–4, 0), (2, 0), (0, –8)
Vertex: (–1, –9)

d.



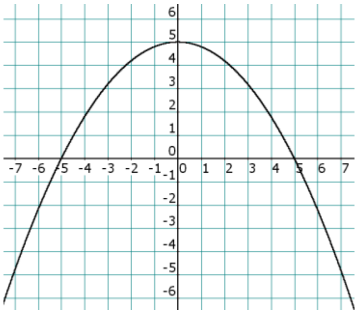
Intercepts:
(2, 0), (0, –4)
Vertex: (2, 0)

e.



Intercepts: (0, 0)
Vertex: (0, 0)

f.

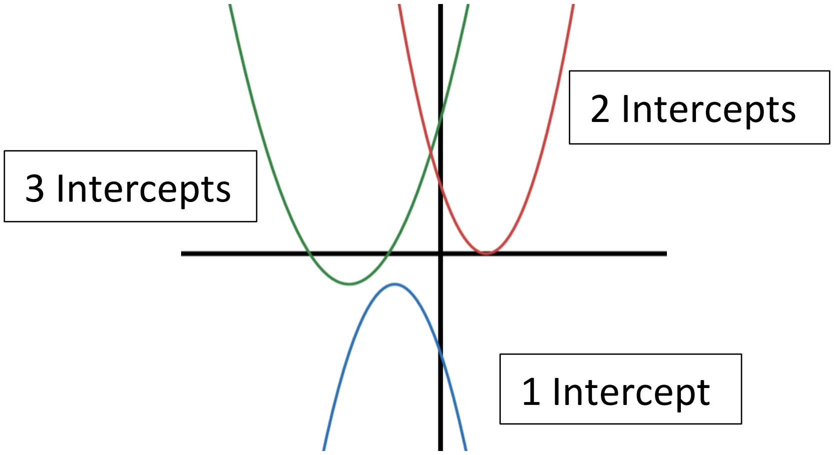


Intercepts:
(–5, 0), (5, 0), (0, 5)
Vertex: (0, 5)

> “It is our choices that show
> what we truly are,
> far more than our abilities.”
>
> – *J.K. Rowling*

CH XX – INTERCEPTS OF A PARABOLA, FACTORABLE

Most LINES have just two intercepts (or just one intercept if it's a horizontal or vertical line that's not one of the axes, or even infinitely many intercepts if the line is one of the axes.) But a PARABOLA may have one intercept, perhaps two intercepts, or possibly three intercepts:



It is impossible for a parabola to have NO intercepts, mainly because in a parabola opening up or down, the *x*-value can be any real number, meaning that the graph goes infinitely to the right and infinitely to the left. If follows that the parabola MUST pass through the *y*-axis somewhere, thus providing at least one intercept.

As in the chapter Graphing Parabolas, we're assuming parabolas that open up or down.

☐ How Do We Find the Intercepts of a Parabola?

Remember the method for finding the intercepts of a line? Well, an intercept is an intercept, so the rules for finding the intercepts of a parabola (or any graph at all!) are identical to the rules we learned before:

> x-intercepts are found by setting **y to 0**.
>
> y-intercepts are found by setting **x to 0**.

Also recall that an intercept is a point in the plane, and should be written as an ordered pair like (2, 0) or (0, –3). Ask your instructor if you need to write your intercepts this way.

EXAMPLE 1: **Find the intercepts of $y = x^2 - x - 6$.**

Solution:

x-intercepts: Setting $y = 0$

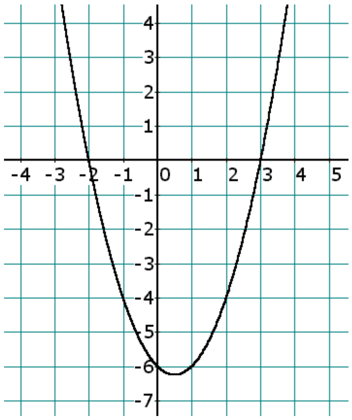
$\Rightarrow \mathbf{0} = x^2 - x - 6$

$\Rightarrow 0 = (x + 2)(x - 3)$

$\Rightarrow x + 2 = 0 \text{ or } x - 3 = 0$

$\Rightarrow x = -2 \text{ or } x = 3$ We conclude that

the x-intercepts are (–2, 0) and (3, 0)



y-intercepts: Setting $x = 0$

$\Rightarrow y = \mathbf{0}^2 - 6(\mathbf{0}) - 6 = -6$

Therefore,

the y-intercept is (0, –6)

EXAMPLE 2: **Find the intercepts of $y = x^2 - 6x + 9$.**

Solution:

x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = x^2 - 6x + 9$$

$$\Rightarrow 0 = (x-3)(x-3)$$

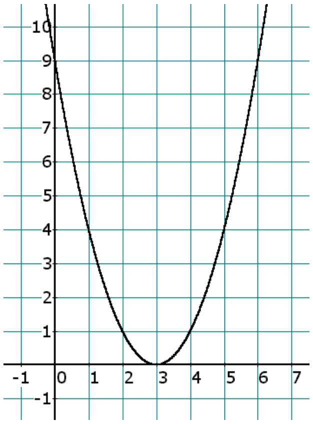
$$\Rightarrow x - 3 = 0$$

$$\Rightarrow x = 3,$$

only one solution for x.

We conclude that

$\boxed{\text{the } x\text{-intercept is } (3, 0)}$



y-intercepts: Setting $x = 0$

$$\Rightarrow y = (\mathbf{0})^2 - 6(\mathbf{0}) + 9 = 9$$

Therefore,

$\boxed{\text{the } y\text{-intercept is } (0, 9)}$

Homework

Find all the **intercepts** of each parabola:

1. $y = x^2 - 25$
2. $y = 2x^2 - 3x - 35$
3. $y = 9x^2 + 6x + 1$
4. $y = 49x^2 - 28x + 4$
5. $y = 9x^2 - 1$
6. $y = 14x^2 - x - 3$
7. $y = 10x^2 + 27x + 14$
8. $y = 15x^2 - 34x + 15$

Solutions

1. (5, 0) (–5, 0) (0, –25)

2. (5, 0) $\left(-\frac{7}{2}, 0\right)$ (0, –35)

3. $\left(-\frac{1}{3}, 0\right)$ (0, 1)

4. $\left(\frac{2}{7}, 0\right)$ (0, 4)

5. $\left(\frac{1}{3}, 0\right)$ $\left(-\frac{1}{3}, 0\right)$ (0, –1)

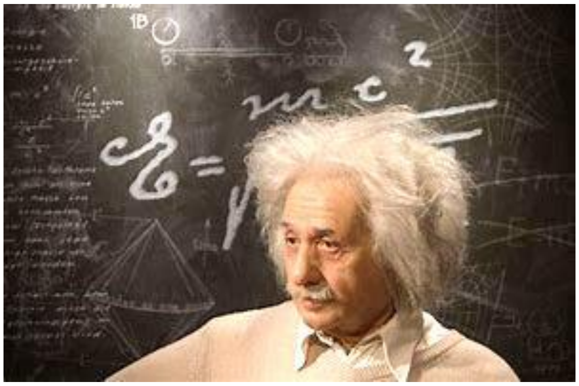
6. $\left(-\frac{3}{7}, 0\right)$ $\left(\frac{1}{2}, 0\right)$ (0, –3)

7. $\left(-\frac{7}{10}, 0\right)$ (–2, 0) (0, 14)

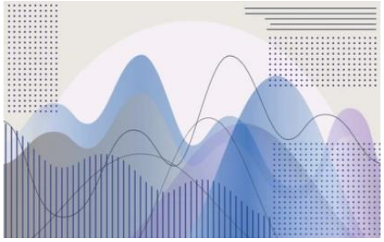
8. $\left(\frac{5}{3}, 0\right)$ $\left(\frac{3}{5}, 0\right)$ (0, 15)

“Any fool can know. The point is to understand.”

– ***Albert Einstein***



CH NN – INTERCEPTS OF A PARABOLA, NON-FACTORABLE



This chapter, like an earlier one, helps you find the **intercepts** of a parabola, but the quadratic equations you will obtain will not be factorable, so another method must be utilized. You could use Completing the Square, but we'll use the Quadratic Formula.

☐ *EXAMPLES*

EXAMPLE 1: **Find the intercepts of $y = -3x^2 + 5x - 1$.**

Solution:

x-intercepts: Setting $y = 0$ turns the parabola equation into

$$\mathbf{0} = -3x^2 + 5x - 1$$

Bringing all the terms to the left side gives us the following quadratic equation in standard form:

$$3x^2 - 5x + 1 = 0$$

The left side of the equation is not factorable (give it a try!), so let's apply the Quadratic Formula, where in this case $a = 3$, $b = -5$, and $c = 1$:

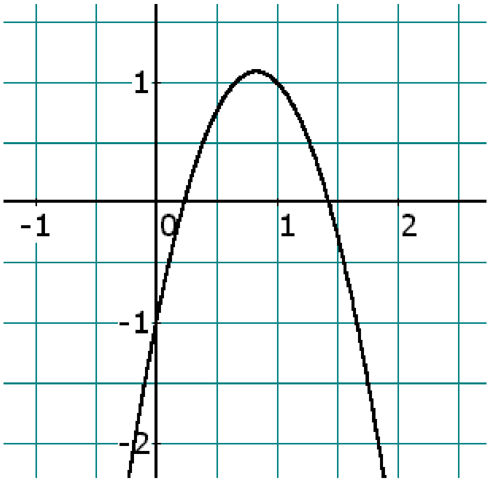
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(1)}}{2(3)}$$
$$= \frac{5 \pm \sqrt{25-12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

These two solutions are certainly correct, and we could even write our two *x*-intercepts like this:

$$\left(\frac{5+\sqrt{13}}{6}, 0\right) \text{ and } \left(\frac{5-\sqrt{13}}{6}, 0\right)$$

However, this form of the *x*-intercepts is not very useful for plotting them on a grid. It's better to use your calculator to convert the two exact radical answers into approximate decimal answers; our *x*-intercepts are roughly

$$\boxed{(1.434, 0) \text{ and } (0.232, 0)}$$



y-intercepts: Setting $x = 0 \Rightarrow$

$$y = -3(\mathbf{0})^2 + 5(\mathbf{0}) - 1 = -1$$

The *y*-intercept is therefore

$$\boxed{(0, -1)}$$

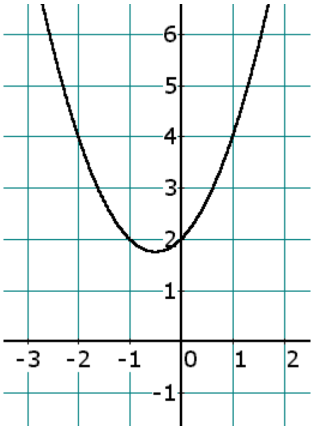
EXAMPLE 2: **Find the intercepts of $y = x^2 + x + 2$.**

Solution: Seems easy enough, but this is a strange one.

x-intercepts: Setting $y = 0$ yields the quadratic equation $x^2 + x + 2 = 0$. First, this quadratic won't factor, but that's okay; we have the Quadratic Formula to rescue us:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

We're in trouble; the –7 in the radical sign indicates that these two solutions for x are <u>not</u> real numbers. This means that wherever these non-real (imaginary) numbers are, they are absolutely <u>not</u> on the x-axis (which is just the set of real numbers).



The conclusion of all this? This parabola has

$$\boxed{\text{no } x\text{-intercept}}$$

<u>y-intercepts:</u> Setting $x = 0$ gives $y = 2$, and so the y-intercept is

$$\boxed{(0, 2)}$$

Homework

1. Find all the **intercepts** of each parabola in exact form — No calculator — no approximations:

a. $y = 2x^2 + 8x + 5$ b. $y = 3x^2 - 6x + 4$

c. $y = x^2 - 13$ d. $y = 2x^2 + 9$

2. Find all the **intercepts** of each parabola, rounded to 3 digits past the point:

a. $y = x^2 + 7x + 1$ b. $y = -2x^2 + 5x + 4$

c. $y = 3x^2 - 6x - 2$ d. $y = 5x^2 + 3x + 1$

Solutions

1. a. $\left(\frac{-4+\sqrt{6}}{2},\ 0\right)$ $\left(\frac{-4-\sqrt{6}}{2},\ 0\right)$ (0, 5)

b. No x-intercepts (0, 4)

c. $\left(\pm\sqrt{13},\ 0\right)$ (0, –13)

d. No x-intercepts (0, 9)

2. a. (–0.146, 0) (–6.854, 0) (0, 1)
b. (3.137, 0) (–0.637, 0) (0, 4)
c. (2.291, 0) (–0.291, 0) (0, –2)
d. No x-intercepts (0, 1)

“Your attitude, not your aptitude, will determine your altitude.”

–Zig Ziglar

CH XX – THE VERTEX OF A PARABOLA

If a parabola opens up, then the ***vertex*** of the parabola is the point at the bottom, the minimum point of the parabola. If the parabola opens down, then the vertex is the top, the maximum point of the parabola.

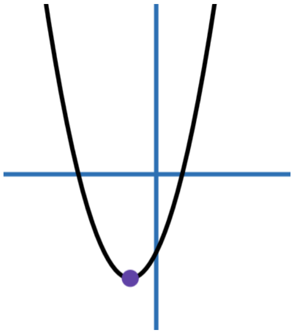
To find the vertex of a parabola in this chapter, we'll need the Midpoint Formula:

If (a, b) and (c, d) are two points in the plane, then the ***midpoint*** *of the line segment connecting those points is*

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

❑ *FINDING THE VERTEX*

The goal is to devise a way to determine the x-coordinate of the **vertex**, the point at the bottom of the graph of the parabola $y = x^2 + 2x - 3$. Once we find the x-coordinate, we can find the y-coordinate by plugging the x-coordinate into the parabola equation.

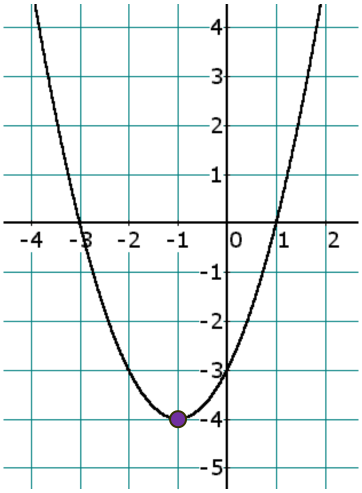


Here's the secret to finding the x-coordinate of the vertex: Go straight up from the vertex to the x-axis. That x-value is <u>midway</u> between the x-values of the two

x-intercepts; in this case it's midway between $x = -3$ and $x = 1$. Recalling the Midpoint Formula reviewed in the Introduction, the x-coordinate of the vertex is

$$x = \frac{-3+1}{2} = \frac{-2}{2} = -\mathbf{1},$$

a fact which should be clear by looking at the graph. Since the parabola's equation is $y = x^2 + 2x - 3$, we find the y-coordinate of the vertex by simply placing $x = -1$ into the equation and finding y:



$$y = x^2 + 2x - 3 = (-\mathbf{1})^2 + 2(-\mathbf{1}) - 3 = 1 - 2 - 3 = -4,$$

which is consistent with the picture. We conclude that the vertex of the parabola is the point **(–1, –4)**.

To find the x-coordinate of the vertex of a parabola, find the <u>average</u> of the x-coordinates of the parabola's x-intercepts.

❑ *EXAMPLES AND FURTHER ANALYSIS*

<u>EXAMPLE 1:</u> **Find the vertex of $y = -2x^2 - 4x + 30$.**

<u>Solution:</u> According to the rule in the box above, we first calculate the x-coordinates of the x-intercepts of the parabola. This, of course, is accomplished by setting y to 0:

$$\mathbf{0} = -2x^2 - 4x + 30$$

$$\Rightarrow \quad 2x^2 + 4x - 30 = 0$$

$\Rightarrow \quad 2(x^2 + 2x - 15) = 0$ (factor out the GCF)

$\Rightarrow \quad 2(x + 5)(x - 3) = 0$ (factor the trinomial)

$\Rightarrow \quad (x + 5)(x - 3) = 0$ (divide both sides by 2)

$\Rightarrow \quad x + 5 = 0 \text{ or } x - 3 = 0$ (set each factor to 0)

$\Rightarrow \quad x = -5 \text{ or } x = 3$ (solve each linear equation)

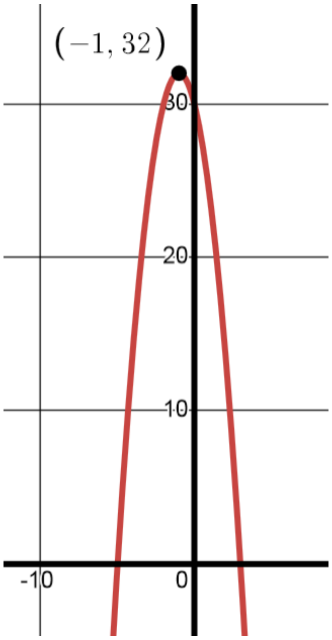
The average of these x-values is $\frac{-5+3}{2} = \frac{-2}{2} = -1$, and so we have $x = -1$ as the x-coordinate of the vertex. The y-value of the vertex is

$$y = -2x^2 - 4x + 30 = -2(\mathbf{-1})^2 - 4(\mathbf{-1}) + 30$$

$$= -2 + 4 + 30 = 32$$

and we conclude that the vertex of the parabola is the point

$$\boxed{(-1, 32)}$$



EXAMPLE 2: **Find the vertex of the parabola $y = x^2 - 6x + 9$.**

Solution: Find the x-intercepts by setting y to 0:

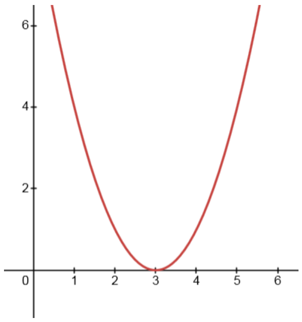
$\mathbf{0} = x^2 - 6x + 9$

$\Rightarrow \quad 0 = (x - 3)(x - 3)$

$\Rightarrow \quad x - 3 = 0 \text{ or } x - 3 = 0$

$\Rightarrow \quad x = 3 \text{ or } x = 3$

There's only one x-intercept, (3, 0), but it did occur twice! So we'll go through the motions and calculate the average of 3 and 3, which is **3**, the x-coordinate of the vertex of the parabola. And the y-coordinate of the vertex is



$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 9 = 9 - 18 + 9 = \mathbf{0}$$

We conclude that the vertex of the parabola is

$$\boxed{(3, 0)}$$

EXAMPLE 3: **Find the vertex of $y = 3x^2 - 7x + 3$.**

Solution: Set y to 0 to calculate the x-intercepts. This will be followed by averaging those x-intercepts to find the x-coordinate of the vertex of the parabola.

$$\mathbf{0} = 3x^2 - 7x + 3$$

Factoring will prove fruitless, so it's time for the Quadratic Formula (we could also Complete the Square) — the values of a, b, and c are 3, –7, and 3:

$$x = \frac{-(\mathbf{-7}) \pm \sqrt{(\mathbf{-7})^2 - 4(\mathbf{3})(\mathbf{3})}}{2(\mathbf{3})} = \frac{7 \pm \sqrt{49 - 36}}{6} = \frac{7 \pm \sqrt{13}}{6}$$

Thus, the x-coordinates of the x-intercepts are

$$\frac{7 + \sqrt{13}}{6} \text{ and } \frac{7 - \sqrt{13}}{6}$$

It is these two numbers that we average to obtain the x-coordinate of the vertex:

$$x = \frac{\frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6}}{2} \quad \text{(average = sum divided by 2)}$$

$$= \frac{\frac{7 + \sqrt{13} + 7 - \sqrt{13}}{6}}{2} \quad \text{(adding the two fractions)}$$

$$= \frac{\frac{14}{6}}{2} = \frac{\frac{7}{3}}{2} = \frac{7}{3} \div 2 = \frac{7}{3} \times \frac{1}{2} = \frac{7}{6}$$

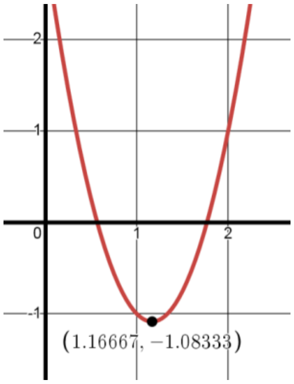
Now we know that the *x*-coordinate of the vertex is $\frac{7}{6}$, and so the *y*-coordinate of the vertex is

$$y = 3\left(\mathbf{\frac{7}{6}}\right)^2 - 7\left(\mathbf{\frac{7}{6}}\right) + 3 = 3\left(\frac{49}{36}\right) - 7\left(\frac{7}{6}\right) + 3 = \frac{49}{12} - \frac{49}{6} + 3$$

$$= \frac{49}{12} - \frac{98}{12} + \frac{36}{12} = -\frac{13}{12}$$

We're finally able to state our conclusion: The vertex of the parabola is

$$\boxed{\left(\frac{7}{6}, -\frac{13}{12}\right)}$$



<u>EXAMPLE 4:</u> **Find the vertex of $y = x^2 + 3x + 4$.**

<u>Solution:</u> Seems innocent enough — what could possibly go wrong? (Ever heard of Murphy's Law?) To find *x*-intercepts, we set *y* to 0 and obtain the quadratic equation

$$\mathbf{0} = x^2 + 3x + 4$$

Since this quadratic is not factorable, we employ the Quadratic Formula:

$$x^2 + 3x + 4 = 0 \qquad a = 1;\ b = 3;\ c = 4$$

$$\Rightarrow \qquad x = \frac{\mathbf{-3} \pm \sqrt{(\mathbf{3})^2 - 4(\mathbf{1})(\mathbf{4})}}{2(\mathbf{1})} = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Gee, these two numbers are not real numbers; you might know that we call them "imaginary" numbers, and that they don't exist anywhere on the *x*-axis (which is an axis of real numbers). We deduce that <u>the parabola has no *x*-intercepts</u>. So how in the heck do we average the *x*-intercepts to find the vertex when they don't even exist? The answer is, trust me! That is, let's calculate the average of these two "phantom" intercepts and see what happens.

There are two issues to consider if we want to calculate the average of two imaginary numbers. First, the two solutions of the quadratic equation above can be written separately as

$$\frac{-3+\sqrt{-7}}{2} \quad \text{and} \quad \frac{-3-\sqrt{-7}}{2}$$

Second, the average of two numbers (even these imaginary numbers!) is found by dividing their sum by 2 — here goes:

$$x = \frac{\dfrac{-3+\sqrt{-7}}{2} + \dfrac{-3-\sqrt{-7}}{2}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\dfrac{-3+\sqrt{-7}-3-\sqrt{-7}}{2}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\dfrac{-6}{2}}{2} \qquad \text{(combine like terms)}$$

$$= \frac{-3}{2} = -\mathbf{\frac{3}{2}}$$

Thus, assuming you still trust me, we will accept this number to be the x-coordinate of the vertex. The y-coordinate would be

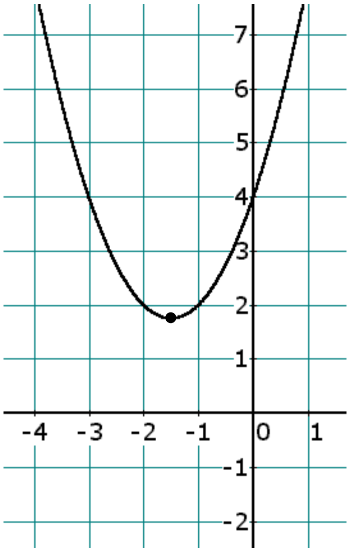
$$y = x^2 + 3x + 4 = \left(-\mathbf{\frac{3}{2}}\right)^2 + 3\left(-\mathbf{\frac{3}{2}}\right) + 4 = \frac{9}{4} - \frac{9}{2} + 4$$

$$= \frac{9}{4} - \frac{18}{4} + \frac{16}{4} = \mathbf{\frac{7}{4}}$$

Our best theory, therefore, is that the vertex of the parabola is

$$\boxed{\left(-\frac{3}{2}, \frac{7}{4}\right)}$$

Let's confirm that this vertex we just calculated of the parabola with no x-intercepts makes some sense by sketching the parabola. It looks pretty reasonable. Notice that the parabola truly has no x-intercepts, just as predicted by the algebra. Moreover, if you look carefully at the vertex of the parabola, you can estimate that the x-coordinate is between -2 and -1, which the number $-\frac{3}{2}$ is. Also, the y-coordinate of the vertex appears to be a little shy of 2, which the number $\frac{7}{4}$ is. I'm convinced — are you?



Homework

1. By averaging the x-coordinates of the x-intercepts, find the **vertex** of each parabola:

a. $y = x^2 + 2x - 48$

b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$

d. $y = 3x^2 - 6x + 4$

☐ *PUTTING IT ALL TOGETHER*

EXAMPLE 5: **Graph $y = x^2 - 6x + 5$ using all the concepts of we've learned about parabolas.**

Solution: The leading coefficient, 1, is positive; thus, the parabola **opens up**.

To find x-intercepts, set y to 0:

$$x^2 - 6x + 5 = \mathbf{0}$$

$$\Rightarrow \quad (x-5)(x-1) = 0$$

$$\Rightarrow \quad x-5 = 0 \text{ or } x-1 = 0$$

$$\Rightarrow \quad x = 5 \text{ or } x = 1$$

Therefore, **the x-intercepts are (1, 0) and (5, 0)**.

As for y-intercepts, we set x to 0:

$$y = \mathbf{0}^2 - 6(\mathbf{0}) + 5 = 5$$

Thus, **the y-intercept is (0, 5)**.

To find the x-coordinate of the vertex we calculate the average of the two x-coordinates of the two x-intercepts:

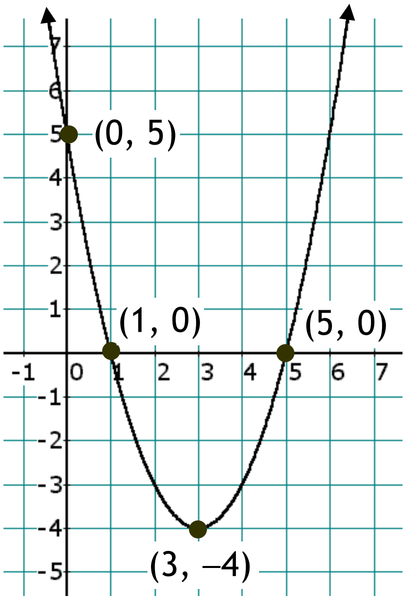
$$x = \frac{1+5}{2} = \frac{6}{2} = 3,$$

which implies that

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 5 = 9 - 18 + 5 = -4$$

So **the vertex of the parabola is (3, –4)**.

We now have our parabola:



Homework

2. Find the y-intercept, the x-intercept(s), and the vertex of $y = 2x^2 + 7x - 5$. Sketch the parabola.

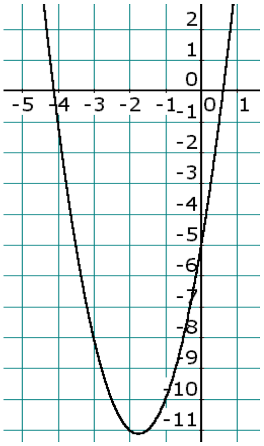
3. Find the y-intercept, the x-intercept(s), and the vertex of $y = -16x^2 + 24x - 9$. Sketch the parabola.

4. Find the y-intercept, the x-intercept(s), and the vertex of $y = x^2 - 3x + 5$. Sketch the parabola.

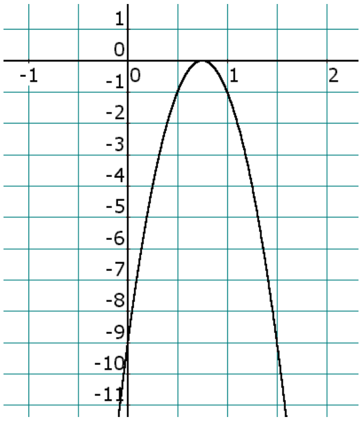
Solutions

1. a. $(-1, -49)$ b. $(-5, 0)$ c. $(-2, -3)$ d. $(1, 1)$

2. y-int: $(0, -5)$ x-int: $\left(\frac{-7+\sqrt{89}}{4}, 0\right)$, $\left(\frac{-7-\sqrt{89}}{4}, 0\right)$ $V\left(-\frac{7}{4}, -\frac{89}{8}\right)$

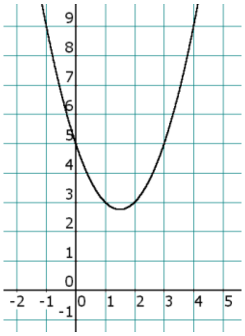


3. y-int: (0, –9) x-int: $\left(\frac{3}{4}, 0\right)$ $V\left(\frac{3}{4}, 0\right)$



Note that the axes are scaled differently. In actuality, the parabola is much skinnier than the one shown.

4. y-int: (0, 5) x-int: None $V\left(\frac{3}{2}, \frac{11}{4}\right)$



“EDUCATION IS THE KEY TO UNLOCK THE GOLDEN DOOR OF FREEDOM.”

– George Washington Carver

CH NN – PARALLEL AND PERPENDICULAR LINES

The chapter title says it all, so let's get to work. First, it's time we revisit our friend, the slope-intercept equation of a line:

$$y = \underset{\text{SLOPE}}{\underset{\uparrow}{m}}x + \underset{y\text{-INTERCEPT}}{\underset{\uparrow}{b}}$$

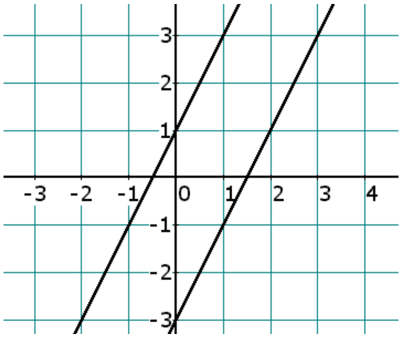
To be precise, *b* is not the *y*-intercept; *b* is the ***y*-coordinate** of the *y*-intercept. The *y*-intercept is properly written (0, *b*).

☐ *PARALLEL LINES*

Let's begin by assuming that throughout this chapter we will never be referring to horizontal or vertical lines — these kinds of lines are covered in detail in the chapter *Special Lines*.

Let's graph the two lines $y = 2x + 1$ and $y = 2x - 3$ on the same grid:

[Is it clear which one is which?]



Notice that the two lines appear to be parallel (and they really are). Now, what do the equations of the two lines have in common? The formulas for the lines show that each line has a slope of **2**. Since slope is a measure of steepness, is it reasonable that if two lines are parallel, then they are equally steep, and therefore they must have the same slope?

Parallel lines have
the **same** slope.

For a simple example: Suppose Line 1 has a slope of –9, and that Line 2 is parallel to Line 1. We can then deduce (conclude by logic) that the slope of Line 2 is also –9, without even graphing it.

EXAMPLE 1: **Find the slope of any line that is *parallel* to the line $7x - 5y = 2$.**

Solution: Any line that is parallel to the line $7x - 5y = 2$ must have the *same* slope as the line $7x - 5y = 2$. So, if we can compute the slope of this line, we will have the slope of any line parallel to it. The easiest way to find the slope of the line is to convert it to $y = mx + b$ form:

$$7x - 5y = 2 \Rightarrow -5y = -7x + 2 \Rightarrow y = \frac{7}{5}x - \frac{2}{5}$$

The slope of the given line is $\frac{7}{5}$, and so we conclude that any line that is parallel to the line $7x - 5y = 2$ must have a slope of

$$\boxed{\frac{7}{5}}$$

NOTE: The examples that follow are solved using the $y = mx + b$ form of a line. However, if you've learned the point-slope form of a line, $y - y_1 = m(x - x_1)$, then you might want to use that form instead.

EXAMPLE 2: **Find the equation of the line that is *parallel* to the line $3x + y = 5$, and that passes through the point (6, 2).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the same as the slope of the given line, since the two lines are parallel. Solving the given line for y yields the line $y = -3x + 5$, whose slope is clearly -3. So the slope of our unknown line is also -3. At this point in the problem we can write our line as

$$y = -3x + b$$

Plugging the given point (6, 2) into this equation allows us to find b:

$$\mathbf{2} = -3(\mathbf{6}) + b$$

$$\Rightarrow \quad 2 = -18 + b$$

$$\Rightarrow \quad 20 = b$$

and we're done; our line is

$$\boxed{y = -3x + 20}$$

Homework

1. a. A given line has a slope of 7. What is the slope of any line that is parallel to the given line?

   b. A given line has a slope of $-\frac{2}{3}$. What is the slope of any line that is parallel to the given line?

2. a. What is the slope of any line that is parallel to the line $y = \frac{3}{4}x - 9$?

   b. What is the slope of any line that is parallel to the line $5x + 2y = 9$?

3. a. Prove that the lines $2x - 4y = 5$ and $3x - 6y = 1$ are parallel.

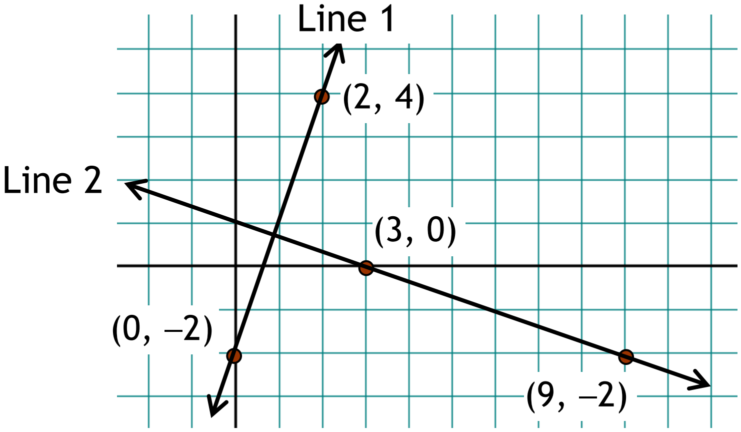
   b. Prove that the lines $3x - y = 4$ and $5x + 2y = 10$ are not parallel.

4. Find the equation of the line that is *parallel* to the given line, and that passes through the given point:

a. $y = 3x + 4$; $(3, -5)$
b. $y = \frac{1}{2}x - 5$; $(-1, 7)$
c. $3x + y = 7$; $(1, 9)$
d. $2x - y = 8$; $(-5, -4)$
e. $2x - 3y = 1$; $(2, -5)$
f. $3x + 4y = 10$; $(8, 0)$

❑ *PERPENDICULAR LINES*

Parallel lines have the same slope — certainly *perpendicular* lines do not. But is there some relationship, some connection, between the slopes of perpendicular lines? Let's see if we can discover one with an example. In the following grid, Line 1 is perpendicular to Line 2, and points have been labeled so that we can easily calculate m_1 and m_2, the slopes of the two lines.



First we compute the slope of Line 1:

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{4-(-2)}{2-0} = \frac{6}{2} = \mathbf{3}$$

Next let's compute the slope of Line 2:

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{0-(-2)}{3-9} = \frac{2}{-6} = -\frac{\mathbf{1}}{\mathbf{3}}$$

There are two things to note regarding these two slopes of the two perpendicular lines. First, one slope is positive while the other is negative. This makes sense because as we move from left to right, Line 1 is increasing while Line 2 is decreasing.

Second, the slope of Line 1, m_1, is kind of a big number (the line's pretty steep), while the slope of Line 2, m_2, (ignoring the minus sign) is a relatively small number (the line's not very steep).

Specifically, the two slopes have opposite signs, and they are also (ignoring the minus sign) *reciprocals* of each other. In other words, when looking at the slopes of two **perpendicular lines**, each of the slopes is the ***opposite reciprocal*** of the other.

Although we have shown this relationship (*opposite reciprocal*) for just this example, it can be proved that this relationship <u>always</u> works for perpendicular lines.

Perpendicular lines have slopes that are **opposite reciprocals** of each other.

For example, if a line has a slope of $\frac{7}{4}$, then any perpendicular line must have a slope of $-\frac{4}{7}$.

And consider the line $y = -5x + 1$. Since its slope is -5, it follows that the slope of any perpendicular line must be $\frac{1}{5}$.

Some books say that the slopes of two perpendicular lines are *negative reciprocals* of each other.

Alternative: We've learned that the slopes of two perpendicular lines are *opposite reciprocals* of each other. But some books say that two lines are perpendicular if their slopes have a product of –1. Do both of these statements mean the same thing? Yes — assume that the product of their slopes is –1:

$$m_1 m_2 = -1$$

Solving for m_1 gives us the equation

$$m_1 = -\frac{1}{m_2},$$

which says that one slope is the *opposite reciprocal* of the other.

Homework

5. A given line has a slope of $-\frac{5}{3}$. What is the slope of any line that is perpendicular to the given line?

6. Prove that the lines $7x - 2y = 10$ and $4x + 14y = 23$ are perpendicular.

7. Prove that the lines $3x + 2y = 10$ and $2x + 3y = 9$ are not perpendicular.

8. Prove that the lines $5x - 3y = 10$ and $5x + 3y = 17$ are not perpendicular.

9. Find the slope of any line that is *perpendicular* to the given line:

   a. $y = -7x + 9$

   b. $y = \frac{5}{4}x + 10$

   c. $2x + 7y = 10$

   d. $3x - 2y = 0$

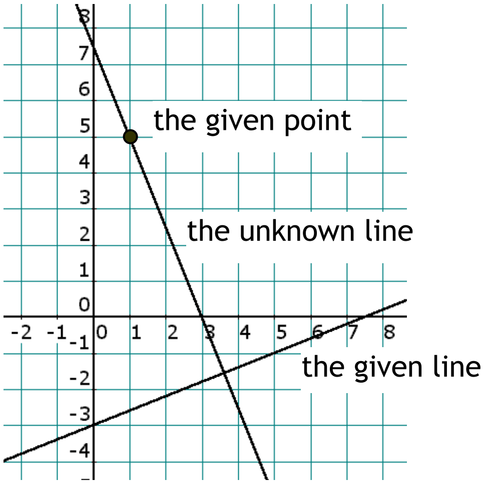
EXAMPLE 3: **Find the equation of the line that is *perpendicular* to the line $2x - 5y = 15$, and that passes through the point (1, 5).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the *opposite reciprocal* of the slope of the given line, since the two lines are perpendicular. To determine the slope of the given line, we solve for y:

$$2x - 5y = 15$$



$$\Rightarrow \quad -5y = -2x + 15$$

$$\Rightarrow \quad \frac{-5y}{-5} = \frac{-2x+15}{-5}$$

$$\Rightarrow \quad y = \frac{2}{5}x - 3$$

telling us that the slope of the given line is $\frac{2}{5}$. So the slope of our unknown line is the opposite reciprocal of that, which is $-\frac{5}{2}$. At this point in the problem we can write our line as

$$y = -\frac{5}{2}x + b$$

Plugging the given point (1, 5) into this equation allows us to find b:

$$\mathbf{5} = -\frac{5}{2}(\mathbf{1}) + b \quad \Rightarrow \quad 5 + \frac{5}{2} = b \quad \Rightarrow \quad b = \frac{15}{2}$$

and we're done; our line is

$$\boxed{y = -\frac{5}{2}x + \frac{15}{2}}$$

Homework

10. Find the equation of the line that is *perpendicular* to the given line, and that passes through the given point:

a. $y = 3x + 4$; (3, –5)

b. $y = \frac{1}{2}x - 5$; (–1, 7)

c. $3x + y = 7$; (1, 9)

d. $2x - y = 8$; (–5, –4)

e. $5x - 3y = 1$; (2, –5)

f. $3x + 4y = 10$; (8, 0)

Review Problems

11. The slopes of two parallel lines are ______.

12. The slopes of two perpendicular lines are ______.

13. The slope of a line is $\frac{5}{7}$. What is the slope of any parallel line?

14. The slope of a line is $-\frac{4}{9}$. What is the slope of any perpendicular line?

15. T/F: The lines $y = 7x - 3$ and $14x - 2y = 22$ are parallel.

16. T/F: The lines $3x - 7y = 1$ and $7x + 3y = 0$ are perpendicular.

17. Find the equation of the line that is parallel to $3x - 7y = 9$ and passes through the point (–3, 10).

18. Find the equation of the line that is perpendicular to $4x - 9y = 11$ and passes through the point (3, –13).

19. Find the equation of the line that is parallel to the line $3x - 4y = 1$ and that passes through the point (–2, –7).

20. Find the equation of the line that is perpendicular to the line $3x - 4y = 1$ and that passes through the point (–2, –7).

21. Which one of the following lines is *parallel* to the line $5x - 3y = 7$?

    a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

    d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

22. Which one of the following lines is *perpendicular* to the line $5x - 3y = 7$?

    a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

    d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

Solutions

1. a. 7 b. $-\frac{2}{3}$

2. a. $\frac{3}{4}$ b. $-\frac{5}{2}$

3. a. Each line has a slope of $\frac{1}{2}$. Same slope $\Rightarrow$ parallel lines.

b. The slopes are 3 and –5/2. Different slopes $\Rightarrow$ non-parallel lines.

4. a. $y = 3x - 14$ b. $y = \frac{1}{2}x + \frac{15}{2}$ c. $y = -3x + 12$

d. $y = 2x + 6$ e. $y = \frac{2}{3}x - \frac{19}{3}$ f. $y = -\frac{3}{4}x + 6$

5. $\frac{3}{5}$

6. The slopes are $\frac{7}{2}$ and $-\frac{2}{7}$, which are opposite reciprocals of each other.

7. The slopes are $-\frac{3}{2}$ and $-\frac{2}{3}$, which are not opposite reciprocals of each other. (They're reciprocals, but not opposites.)

8. The slopes are $\frac{5}{3}$ and $-\frac{5}{3}$, which are not opposite reciprocals of each other. (They're opposites, but not reciprocals.)

9. a. $\frac{1}{7}$ b. $-\frac{4}{5}$ c. $\frac{7}{2}$ d. $-\frac{2}{3}$

10. a. $y = -\frac{1}{3}x - 4$ b. $y = -2x + 5$ c. $y = \frac{1}{3}x + \frac{26}{3}$

d. $y = -\frac{1}{2}x - \frac{13}{2}$ e. $y = -\frac{3}{5}x - \frac{19}{5}$ f. $y = \frac{4}{3}x - \frac{32}{3}$

11. equal **12**. opposite reciprocals

13. $\frac{5}{7}$ **14**. $\frac{9}{4}$ **15**. T **16**. T

17. $y = \frac{3}{7}x + \frac{79}{7}$

18. $y = -\frac{9}{4}x - \frac{25}{4}$

19. $y = \frac{3}{4}x - \frac{11}{2}$

20. $y = -\frac{4}{3}x - \frac{29}{3}$

21. c.

22. b.

> “Upon the subject of education … I can only say that I view it as the most important subject which we as a people may be engaged in.”

> – *Abraham Lincoln*

CH NN – DIVIDING POLYNOMIALS

❑ INTRODUCTION

First we need the proper terminology. When written as a fraction, a division problem has two parts:

$$\frac{\text{dividend}}{\text{divisor}}$$

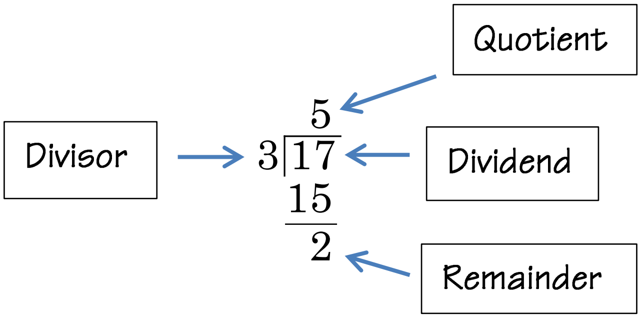
When written in the standard “long division” format, we write

$$\text{divisor}\overline{)\text{dividend}}$$

And we can use the “ $\div$ ” symbol for division:

$$\text{dividend} \div \text{divisor}$$

The result of dividing is called the ***quotient***, and the leftover is called the ***remainder***. For example,



We can then write the answer as $5+\frac{2}{3}$ $\left(\text{quotient} + \frac{\text{remainder}}{\text{divisor}}\right)$, or as the mixed number $5\frac{2}{3}$ when we’re dealing with numbers.

☐ DIVIDING A POLYNOMIAL BY A MONOMIAL

Just as $\frac{1}{7}+\frac{3}{7} = \frac{4}{7}$, we can work the problem $\frac{a}{b}+\frac{c}{b}$ by adding the numerators, and placing that sum over the common denominator *b*:

$$\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

By reversing this reasoning, we can take the fraction $\frac{a+c}{b}$ and, if we like, *split* it into the sum of two fractions:

$$\frac{a+c}{b} = \frac{a}{b}+\frac{c}{b}$$

This is the trick we need to divide a polynomial by a monomial.

EXAMPLE 1: **Divide:** $\boldsymbol{\frac{ab^2+a^2b+ab}{ab}}$

Solution: Separate the fraction into three separate fractions, keeping the denominator for all three fractions:

$$\frac{ab^2}{ab} + \frac{a^2b}{ab} + \frac{ab}{ab}$$

and then simplify (reduce) each fraction:

$$\boxed{b+a+1}$$

EXAMPLE 2: **Divide:** $\boldsymbol{\frac{12x^2y^3-8x^3y^2+7xy^3}{2x^2y}}$

Solution: Split the fraction into three separate fractions:

$$\frac{12x^2y^3}{2x^2y} - \frac{8x^3y^2}{2x^2y} + \frac{7xy^3}{2x^2y}$$

and then simplify (reduce) each fraction:

$$\boxed{6y^2 - 4xy + \frac{7y^2}{2x}}$$

Homework

1. Perform each division problem, where the divisor (the bottom) is a monomial:

a. $\dfrac{x^3 - x^2 + x}{x}$ b. $\dfrac{14xy + 21x^2y - 28xy^2}{7xy}$ c. $\dfrac{x^2 + 3x + 1}{x}$

d. $\dfrac{a+b}{b}$ e. $\dfrac{x-y}{y}$ f. $\dfrac{ax+bx}{x}$

❑ *DIVIDING A POLYNOMIAL BY A POLYNOMIAL*

Think back when you were a kid and learned long division of numbers. Though I've seen different ways of doing this, the following method works well in algebraic long division. It boils down to a 4-step process, a process that is repeated until the problem is finished:

1. Divide the divisor into the first part of the dividend.
2. Multiply the part of the quotient calculated in step 1 by the divisor.
3. Subtract.
4. Bring down the next digit.

And then repeat steps 1–4 as many times as necessary.

We use the same process for polynomial long division in algebra.

EXAMPLE 3: **Perform the long division:** $\boldsymbol{\dfrac{3x^3-5x-2}{x-1}}$

Solution: The first step is to fill in the missing term in the dividend. Since there is no x^2 term, we insert $\mathbf{0}\boldsymbol{x}^{\mathbf{2}}$ as a "placeholder" between the cubic term and the linear term, giving us a dividend of $3x^3\ \mathbf{+\ 0}\boldsymbol{x}^{\mathbf{2}} - 5x - 2$. So our long division problem is

$$x-1\,\overline{\big)\,3x^3+0x^2-5x-2}$$

1. Divide x into $3x^3$; it goes in $3x^2$ times (since $3x^2\cdot x\ =\ 3x^3$):

$$\begin{array}{rl} & \boldsymbol{3x^2} \\ \boldsymbol{x}-1\big) & \overline{\boldsymbol{3x^3}+0x^2-5x-2} \end{array}$$

2. Multiply $3x^2$ by the divisor, $x-1$:

$$\begin{array}{rl} & \boldsymbol{3x^2} \\ \boldsymbol{x-1}\big) & \overline{3x^3+0x^2-5x-2} \\ & \underline{\boldsymbol{3x^3-3x^2}} \end{array}$$

3. Subtract; $3x^3 - 3x^3\ =\ 0$; $0x^2 - (-3x^2)\ =\ 3x^2$:

$$\begin{array}{rl} & 3x^2 \\ x-1\big) & \overline{\boldsymbol{3x^3}+\boldsymbol{0x^2}-5x-2} \\ & \underline{-\,(\boldsymbol{3x^3-3x^2})} \\ & \quad\mathbf{0}\quad+\boldsymbol{3x^2} \end{array}$$

4. Bring down the next term, $-5x$:

$$\begin{array}{rl} & 3x^2 \\ x-1\big) & \overline{3x^3+0x^2-\boldsymbol{5x}-2} \\ & \underline{-\,(3x^3-3x^2)}\ \big\downarrow \\ & \quad 0\ +3x^2-\boldsymbol{5x} \end{array}$$

1. And repeat: <u>Divide</u> x into $3x^2$:

$$\begin{array}{rl} & \;\; 3x^2 \;\; + \boldsymbol{3x} \\ \boldsymbol{x} - 1 & \big|\overline{3x^3 + 0x^2 - 5x - 2} \\ & \underline{-\,(3x^3 - 3x^2)} \\ & \quad 0 \;\; + \boldsymbol{3x^2} - 5x \end{array}$$

2. <u>Multiply</u> $3x$ by $x - 1$, the divisor:

$$\begin{array}{rl} & \;\; 3x^2 \;\; + \boldsymbol{3x} \\ \boldsymbol{x - 1} & \big|\overline{3x^3 + 0x^2 - 5x - 2} \\ & \underline{-\,(3x^3 - 3x^2)} \\ & \quad 0 \;\; + 3x^2 - 5x \\ & \qquad\;\; \boldsymbol{3x^2} - \boldsymbol{3x} \end{array}$$

3. <u>Subtract</u>; $3x^2 - 3x^2 = 0; -5x - (-3x) = -2x$:

$$\begin{array}{rl} & \;\; 3x^2 \;\; + 3x \\ x - 1 & \big|\overline{3x^3 + 0x^2 - 5x - 2} \\ & \underline{-\,(3x^3 - 3x^2)} \\ & \quad 0 \;\; + \boldsymbol{3x^2} - \boldsymbol{5x} \\ & \qquad \underline{-(\boldsymbol{3x^2} - \boldsymbol{3x})} \\ & \qquad\quad \mathbf{0} \;\; - \boldsymbol{2x} \end{array}$$

4. <u>Bring down</u> the next (and last) term, -2:

$$\begin{array}{rl} & \;\; 3x^2 \;\; + 3x \\ x - 1 & \big|\overline{3x^3 + 0x^2 - 5x - \mathbf{2}} \\ & \underline{-\,(3x^3 - 3x^2)} \\ & \quad 0 \;\; + 3x^2 - 5x \\ & \qquad \underline{-(3x^2 - 3x)} \downarrow \\ & \qquad\quad 0 \;\; - 2x - \mathbf{2} \end{array}$$

1. <u>Divide</u> x into $-2x$:

$$\begin{array}{r l}
 & 3x^2 + 3x - \mathbf{2} \\
\boldsymbol{x} - 1 & \big)\overline{3x^3 + 0x^2 - 5x - 2} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & 0 + 3x^2 - 5x \\
 & \quad \underline{-(3x^2 - 3x)} \\
 & \qquad 0 - \boldsymbol{2x} - 2
\end{array}$$

2. <u>Multiply</u> -2 by $x - 1$:

$$\begin{array}{r l}
 & 3x^2 + 3x - \mathbf{2} \\
\boldsymbol{x} - \mathbf{1} & \big)\overline{3x^3 + 0x^2 - 5x - 2} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & 0 + 3x^2 - 5x \\
 & \quad \underline{-(3x^2 - 3x)} \\
 & \qquad 0 - 2x - 2 \\
 & \qquad\quad \mathbf{-\,2\boldsymbol{x} + 2}
\end{array}$$

3. <u>Subtract</u>; $-2x - (-2x) = 0$; $-2 - (+2) = -4$

$$\begin{array}{r l}
 & 3x^2 + 3x - 2 \\
x - 1 & \big)\overline{3x^3 + 0x^2 - 5x - 2} \\
 & \underline{-(3x^3 - 3x^2)} \\
 & 0 + 3x^2 - 5x \\
 & \quad \underline{-(3x^2 - 3x)} \\
 & \qquad 0 - \mathbf{2\boldsymbol{x} - 2} \\
 & \qquad \underline{\mathbf{-(-2\boldsymbol{x} + 2)}} \\
 & \qquad\quad \mathbf{0 \ \ -4}
\end{array}$$

There are no terms left to bring down in the dividend, so we write the remainder (the –4) over the divisor and add it to the quotient. The final answer to the long division problem is

$$\boxed{3x^2 + 3x - 2 + \frac{-4}{x - 1}}$$

Homework

2. Perform each polynomial long division problem, expressing any remainder as a fraction added to the quotient:

a. $\frac{x^2+5x+6}{x+3}$ b. $\frac{x^2-9}{x-3}$ c. $\frac{x^2+2x+1}{x+1}$

d. $\frac{n^2+n-4}{n+5}$ e. $\frac{2a^2-5a+2}{a+3}$ f. $\frac{3w^2+10}{w+5}$

g. $\frac{6b^2+b-15}{2b+3}$ h. $\frac{3y^2-9}{y+5}$ i. $\frac{10x^2+3x-7}{2x-1}$

j. $\frac{x^3+1}{x+1}$ Hint: $x^3 + 1 = x^3 + 0x^2 + 0x + 1$

k. $\frac{n^3-8}{n-2}$ l. $\frac{a^3+27}{a^2-3a+9}$

3. Perform each polynomial long division problem (Hint: there is no remainder):

a. $\frac{40x^3+97x^2+60x+27}{5x+9}$ b. $\frac{8w^3+22w^2+13w+2}{2w^2+5w+2}$

c. $\frac{40r^3-4r^2-7r-3}{8r^2+4r+1}$ d. $\frac{63m^3+43m^2+13m+1}{7m^2+4m+1}$

Review Problems

4. Divide: $\dfrac{4x^3-8x^2+6x-10}{4x^2}$

5. Divide: $\dfrac{x^2+9}{x-5}$

6. Divide: $\dfrac{x^3-3x+8}{x+3}$

7. Divide: $\dfrac{x^4-1}{x+1}$

8. Divide: $\dfrac{n^3+8}{n+2}$

Solutions

1. a. $x^2 - x + 1$
b. $2 + 3x - 4y$
c. $x+3+\dfrac{1}{x}$
d. $\dfrac{a}{b}+1$
e. $\dfrac{x}{y}-1$
f. $a + b$

2. a. $x + 2$
b. $x + 3$
c. $x + 1$
d. $n-4+\dfrac{16}{n+5}$
e. $2a-11+\dfrac{35}{a+3}$
f. $3w-15+\dfrac{85}{w+5}$
g. $3b-4+\dfrac{-3}{2b+3}$
h. $3y-15+\dfrac{66}{y+5}$
i. $5x+4+\dfrac{-3}{2x-1}$
j. $x^2 - x + 1$
k. $n^2 + 2n + 4$
l. $a + 3$

3. a. $8x^2 + 5x + 3$
b. $4w + 1$
c. $5r - 3$
d. $9m + 1$

4. $x-2+\frac{3}{2x}-\frac{5}{2x^2}$

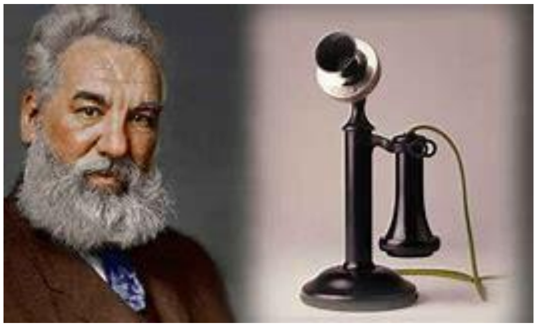
5. $x+5+\frac{34}{x-5}$

6. $x^2-3x+6+\frac{-10}{x+3}$

7. x^3-x^2+x-1

8. n^2-2n+4

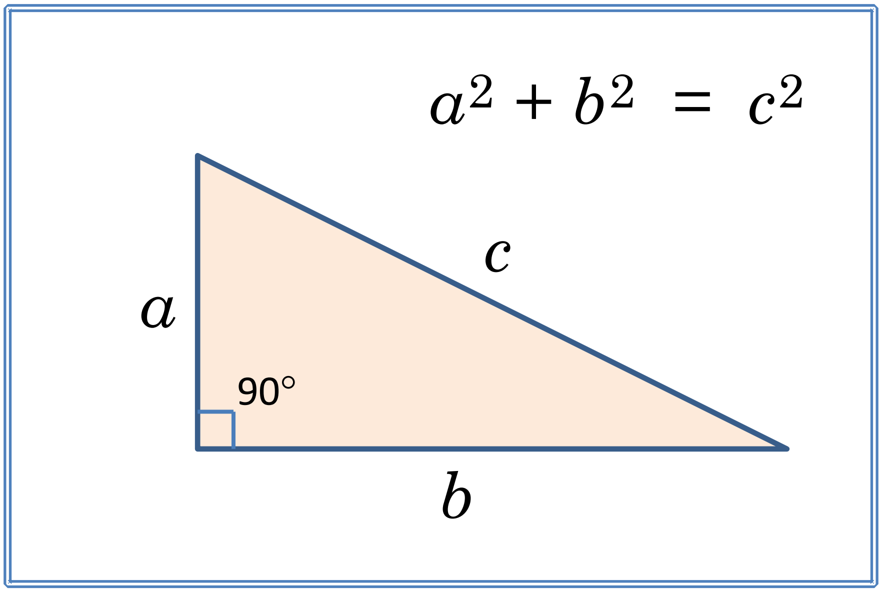
“When one door closes, another opens; but we often look so long and so regretfully upon the closed door that we do not see the one which has opened for us.”



- Alexander Graham Bell

CH XX –THE PYTHAGOREAN THEOREM, FINDING A LEG

❑ RECALLING THE PYTHAGOREAN THEOREM



The 90° angle is called the **right** angle of the right triangle. The other two angles of the right triangle are called the **acute** angles, because each of them must be less than 90°. The **legs** are *a* and *b*, and the **hypotenuse** is *c*.

EXAMPLE 1: **One leg of a right triangle is 7, while the hypotenuse is 25. Find the length of the other leg.**

Solution: Prior to this chapter we were always given the two legs and asked to find the hypotenuse. This problem throws us a curve, but we now have the skills needed to solve it. Since the sum of the squares of the legs must equal the square of the hypotenuse, we start with the Pythagorean Theorem,

$$a^2 + b^2 = c^2$$ *a* and *b* are legs; *c* is the hypotenuse

Putting in a leg of 7 and a hypotenuse of 25, the formula becomes

$$a^2 + \mathbf{7}^2 = \mathbf{25}^2$$ (the leg can be either a or b)

$$\Rightarrow \quad a^2 + 49 = 625$$ (square the numbers)

$$\Rightarrow \quad a^2 + 49 - \mathbf{49} = 625 - \mathbf{49}$$ (subtract 49 from each side)

$$\Rightarrow \quad a^2 = 576$$ (simplify each side)

$$\Rightarrow \quad a = 24$$ (calculator or guessing)

We conclude that the other leg has a length of $\boxed{24}$

<u>Check:</u> If we're claiming that a right triangle can have legs of 24 and 7 and a hypotenuse of 25, then these three values had better satisfy the Pythagorean Theorem. Let's see if they do:

$$(\text{leg})^2 + (\text{leg})^2 \overset{?}{=} (\text{hypotenuse})^2$$

$$24^2 + 7^2 \overset{?}{=} 25^2$$

$$576 + 49 \overset{?}{=} 625$$

$$625 = 625 \checkmark$$

<u>EXAMPLE 2:</u> Find the hypotenuse of a right triangle whose legs are 5 and 7.

<u>Solution:</u>

$$a^2 + b^2 = c^2$$ (the Pythagorean Theorem)

$$\mathbf{5}^2 + \mathbf{7}^2 = c^2$$ (substitute the known values)

$$25 + 49 = c^2$$ (square each leg)

$$74 = c^2$$ (simplify)

Is there a whole number whose square is 74? No, because $8^2 = 64$, which is smaller than 74, while $9^2 = 81$, which is bigger than 74. We see, therefore, that the solution is somewhere between the consecutive numbers 8 and 9; but where between 8 and 9?

$8^2 = 64$ ✓
what^2 = 74 ??
$9^2 = 81$ ✓

Finding a number whose square is 74 is the kind of problem that has plagued and enticed mathematicians, scientists, and philosophers for literally thousands of years. They'd really be mad if they knew that we can find an excellent approximation of this number using a cheap calculator. Enter the number 74 followed by the *square root* key (or the other way around, depending on the calculator), which is labeled something like $\sqrt{x}$. Thus, the hypotenuse is $\sqrt{74}$ (read: *the positive square root of 74*), and your calculator should have the result 8.602325267, or something close to that.

But your calculator doesn't tell the whole story. The fact is, the square root of 74 has an infinite number of digits following the decimal point, and they never have a repeating pattern (very similar to the number π). Thus, we'll have to round off the answer to whatever's appropriate for the problem. For this problem, we'll round to the third digit past the point.

The number $\sqrt{74}$ is called **irrational**, due to its infinite, non-repeating decimal expansion.

The hypotenuse is 8.602

Notice that $8.602^2 = 73.994404$, which is quite close to 74.

EXAMPLE 3: **The hypotenuse of a right triangle is 15 and one of its legs is 4. Find the length of the other leg (rounded to 3 digits).**

Solution: Let's get right to it:

$$a^2 + b^2 = c^2$$

$$\Rightarrow \mathbf{4}^2 + b^2 = \mathbf{15}^2$$

$$\Rightarrow 16 + b^2 = 225$$

$$\Rightarrow b^2 = 209$$

$$\Rightarrow b = \boxed{14.457} \text{ (approximately)}$$

This result was found using a calculator; if no calculator is allowed, then you'd write the answer like this:

$\boxed{\text{The leg is between 14 and 15.}}$

Homework

1. In each problem, one leg and the hypotenuse of a right triangle are given (l = leg and h = hypotenuse). Find the other leg — NO calculator.

a. $l = 3$; $h = 5$

b. $l = 5$; $h = 13$

c. $l = 6$; $h = 10$

d. $l = 9$; $h = 41$

e. $l = 12$; $h = 13$

f. $l = 24$; $h = 25$

2. In each problem, one leg and the hypotenuse of a right triangle are given (l = leg and h = hypotenuse). Use your calculator to find the other leg, rounded to 3 digits. If no calculator is allowed, tell what two consecutive whole numbers the answer is between.

a. $l = 3$; $h = 10$	b. $l = 2$; $h = 9$	c. $l = 5$; $h = 6$
d. $l = 4$; $h = 20$	e. $l = 7$; $h = 19$	f. $l = 13$; $h = 55$

3. A triangle has sides of 6, 10, and 8. Prove that it's a *right triangle*. Hint: See the Check in Example 1.

4. A triangle has sides of 5, 9, and 7. Prove that it's not a *right triangle*.

   [Note: Whether you're trying to prove that a triangle is – or is not – a right triangle, logic dictates that you must assume that the longest of the three given sides is the hypotenuse.]

Review Problems

5. The hypotenuse of a right triangle is 65 and one of its legs is 60. Find the other leg.
6. Use a calculator (if allowed) to find the leg of a right triangle if its hypotenuse is 89 and one of its legs is 1. Round to the 3rd digit.
7. Verify that a triangle with sides 24, 10, and 26 is a right triangle.
8. Is a triangle with sides 15, 12, and 8 a right triangle?

Solutions

1. a. 4 b. 12 c. 8 d. 40
e. 5 f. 7

2. a. 9.539 b. 8.775 c. 3.317 d. 19.596
e. 17.664 f. 53.442

3. Verify that $6^2 + 8^2 = 10^2$.

4. Show that $5^2 + 7^2 \neq 9^2$. Notice that even though the 9 was listed second in the list of the three sides of the triangle, we used the 9 as the *potential* hypotenuse.

5. 25 **6.** 88.99

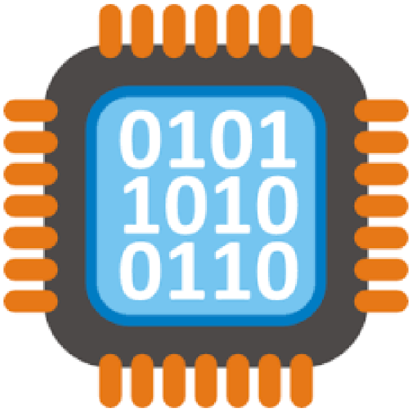
7. Verify that $10^2 + 24^2 = 26^2$.

8. No; be sure you assume that the hypotenuse would be the 15.

> ***"It is possible to store the mind with a million facts and still be entirely uneducated."***
>
> **– Alec Bourne, M.D.**

CH XX – THE PYTHAGOREAN THEOREM WITH RADICALS

The last time we studied the Pythagorean Theorem we may have used our calculator to round off square roots that didn't come out whole numbers. But rounding off square roots is not always a good idea in the real world.



After we've done some applied mathematics, we may give the results to an engineer. Suppose we've rounded to the hundredths place (good enough for business) but the engineer is designing the Core 9 Quad microprocessor, and she needs all the digits of the calculations out to the billionths place. We haven't done her any good at all by rounding off for her. The safest thing to do is to give her the exact results and let her round off according to her requirements.

We're now going to work with the Pythagorean Theorem again, but this time we're going to leave our answers exact. To this end, we need to review the key concept regarding square roots.

☐ *AN IMPORTANT FACT ABOUT SQUARE ROOTS*

If we square the square root of "something" (that's not negative), we get the "something":

$$(\sqrt{25})^2 = 5^2 = 25$$

$$(\sqrt{34})^2 \approx (5.830951895)^2 \approx 34$$

And we can write the general formula:

$$\boxed{\left(\sqrt{x}\right)^2 = x}$$

Squaring Undoes Square-Rooting

We assume $x \geq 0$, lest we take the square root of a negative number, resulting in a non-real number.

Another skill we'll need for this chapter is the ability to simplify an expression like $(5\sqrt{3})^2$. Recall one of our five laws of exponents: $(ab)^2 = a^2b^2$.

This rule means that we can simplify $(5\sqrt{3})^2$ as follows:

$$(5\sqrt{3})^2 = 5^2(\sqrt{3})^2 = 25 \cdot 3 = \mathbf{75}$$

Homework

1. While it's the case that $(ab)^2 = a^2b^2$, is this rule true for addition? That is, does $(a + b)^2 = a^2 + b^2$?

2. Simplify each expression:

a. $(\sqrt{7})^2$ b. $(\sqrt{29})^2$ c. $(\sqrt{233})^2$ d. $(\sqrt{101})^2$

e. $(2\sqrt{3})^2$ f. $(5\sqrt{5})^2$ g. $(9\sqrt{2})^2$ h. $(12\sqrt{7})^2$

3. Example: $\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = \mathbf{10\sqrt{2}}$

Simplify each radical:

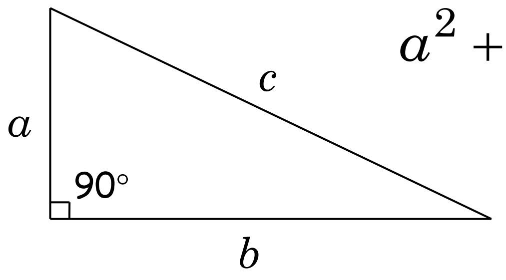
a. $\sqrt{144}$ b. $\sqrt{225}$ c. $\sqrt{1}$ d. $\sqrt{0}$

e. $\sqrt{-49}$ f. $\sqrt{14}$ g. $\sqrt{200}$ h. $\sqrt{45}$

i. $\sqrt{98}$ j. $\sqrt{841}$ k. $\sqrt{48}$ l. $\sqrt{112}$

☐ *SOLVING RIGHT TRIANGLES*

Recall the Pythagorean Theorem:



$$a^2 + b^2 = c^2$$

<u>EXAMPLE 1</u>: **The legs are 6 and 10. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{6}^2 + \mathbf{10}^2 = c^2 \Rightarrow 36 + 100 = c^2$$
$$\Rightarrow c^2 = 136 \Rightarrow c = \sqrt{136} = \sqrt{4 \cdot 34} = \mathbf{2\sqrt{34}}$$

Notice that $c^2 = 136$ is a quadratic equation, and therefore probably has two solutions; in fact, the solutions are $\pm\sqrt{136}$, or $\pm 2\sqrt{34}$. But in this problem we're talking about the hypotenuse of a triangle, whose length must be positive. So we immediately discard the negative solution and retain just the positive one.

EXAMPLE 2: **One leg is 20 and the hypotenuse is 30. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{20}^2 + b^2 = \mathbf{30}^2$$

$$\Rightarrow 400 + b^2 = 900 \Rightarrow b^2 = 500$$

$$\Rightarrow b = \sqrt{500} = \sqrt{100 \cdot 5} = \mathbf{10\sqrt{5}}$$

EXAMPLE 3: **The legs are $2\sqrt{7}$ and $3\sqrt{13}$. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow (\mathbf{2\sqrt{7}})^2 + (\mathbf{3\sqrt{13}})^2 = c^2$$

$$\Rightarrow 28 + 117 = c^2 \Rightarrow c^2 = 145 \Rightarrow c = \mathbf{\sqrt{145}}$$

EXAMPLE 4: **The hypotenuse is $\sqrt{72}$ and one of the legs is 8. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow a^2 + \mathbf{8}^2 = (\mathbf{\sqrt{72}})^2 \Rightarrow a^2 + 64 = 72$$

$$\Rightarrow a^2 = 8 \Rightarrow a = \sqrt{8} = \sqrt{4 \cdot 2} = \mathbf{2\sqrt{2}}$$

EXAMPLE 5: **One of the legs is $3\sqrt{2}$ and the hypotenuse is $\sqrt{42}$. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow a^2 + (\mathbf{3\sqrt{2}})^2 = (\mathbf{\sqrt{42}})^2$$

$$\Rightarrow a^2 + 18 = 42 \Rightarrow a^2 = 24$$

$$\Rightarrow a = \sqrt{24} = \mathbf{2\sqrt{6}}$$

Homework

Calculate the EXACT answer to each Pythagorean problem:

4. The legs of a right triangle are 8 and 12. Find the hypotenuse.
5. The legs of a right triangle are 6 and 9. Find the hypotenuse.
6. Find the hypotenuse of a right triangle given that its legs have lengths of 5 and 7.
7. One leg of a right triangle is 15 and the hypotenuse is 20. Find the other leg.
8. The hypotenuse of a right triangle is 18 and one of its legs is 10. Find the other leg.
9. One leg of a right triangle is 5 and the hypotenuse is $\sqrt{43}$. Find the other leg.
10. Find the missing leg of a right triangle if the hypotenuse is $\sqrt{18}$ and one of its legs is 3.
11. The legs of a right triangle are $2\sqrt{3}$ and $3\sqrt{2}$. Find the hypotenuse.
12. The hypotenuse of a right triangle is $5\sqrt{7}$ and one of its legs is $2\sqrt{3}$. Find the length of the other leg.
13. A leg of a right triangle is $2\sqrt{8}$ and its hypotenuse is 10. Find the length of the other leg.

14. The leg of a right triangle is 5 and its hypotenuse is 4. Find the other leg.

Review Problems

15. In each problem, two of the sides of a right triangle are given (l = leg and h = hypotenuse). Find the third side.

a. $l = 7;\ l = 10$

b. $l = 1;\ l = 3$

c. $l = 4;\ l = 6$

d. $l = \sqrt{8};\ l = 2\sqrt{2}$

e. $l = 4;\ l = \sqrt{10}$

f. $l = \sqrt{2};\ l = \sqrt{14}$

g. $l = 5;\ h = 12$

h. $l = \sqrt{5};\ h = 2\sqrt{3}$

i. $l = 6;\ h = 5\sqrt{2}$

j. $l = \sqrt{7};\ h = \sqrt{31}$

k. $l = \sqrt{2};\ h = 2\sqrt{5}$

l. $l = \sqrt{5};\ h = 4\sqrt{5}$

m. $l = \sqrt{2};\ l = \sqrt{7}$

n. $l = 7;\ h = 8$

o. $l = \sqrt{5};\ h = \sqrt{13}$

p. $l = 5;\ h = \sqrt{26}$

q. $l = \sqrt{10};\ h = \sqrt{41}$

r. $l = \sqrt{11};\ h = 8$

s. $l = \sqrt{13};\ l = \sqrt{13}$

t. $l = 5;\ l = \sqrt{6}$

u. $l = 4;\ l = 5$

v. $l = 1;\ h = 2$

w. $l = 1;\ l = 1$

x. $l = 2\sqrt{2};\ h = 2\sqrt{5}$

y. $l = \sqrt{5};\ h = \sqrt{21}$

z. $l = 3\sqrt{2};\ h = 20$

Solutions

1. NO! You should know quite well that

$$(a+b)^2 = (a+b)(a+b) = a^2+ab+ba+b^2 = \underline{a^2+2ab+b^2}$$

2. a. 7 b. 29 c. 233 d. 101
e. 12 f. 125 g. 162 h. 1008

3. a. 12 b. 15 c. 1 d. 0
e. Not a Real Number f. $\sqrt{14}$ g. $10\sqrt{2}$ h. $3\sqrt{5}$
i. $7\sqrt{2}$ j. 29 k. $4\sqrt{3}$ l. $4\sqrt{7}$

4. $4\sqrt{13}$ **5**. $3\sqrt{13}$ **6**. $\sqrt{74}$ **7**. $5\sqrt{7}$ **8**. $4\sqrt{14}$ **9**. $3\sqrt{2}$

10. 3 **11**. $\sqrt{30}$ **12**. $\sqrt{163}$ **13**. $2\sqrt{17}$

14. This scenario is impossible, since the hypotenuse of a right triangle must be longer than either of its legs. Here's a mathematical proof:

$$a^2+b^2 = c^2$$

$$\Rightarrow\ a^2+5^2 = 4^2$$ (presuming that the leg is 5 and the hypotenuse is 4)

$$\Rightarrow\ a^2+25 = 16$$

$$\Rightarrow\ a^2 = -9$$ (subtract 25 from each side of the equation)

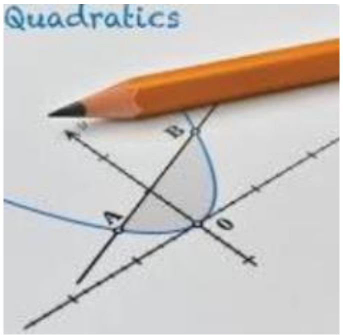
$$\Rightarrow\ a = \sqrt{-9},$$ which is not a number in this class. And even if it becomes a number in a later class, it still couldn't be considered the length of a side of a triangle.

15. a. $\sqrt{149}$ b. $\sqrt{10}$ c. $2\sqrt{13}$ d. 4 e. $\sqrt{26}$
f. 4 g. $\sqrt{119}$ h. $\sqrt{7}$ i. $\sqrt{14}$ j. $2\sqrt{6}$
k. $3\sqrt{2}$ l. $5\sqrt{3}$ m. 3 n. $\sqrt{15}$ o. $2\sqrt{2}$
p. 1 q. $\sqrt{31}$ r. $\sqrt{53}$ s. $\sqrt{26}$ t. $\sqrt{31}$
u. 3 v. $\sqrt{3}$ w. $\sqrt{2}$ x. $2\sqrt{3}$ y. 4
z. $\sqrt{382}$

> *“He who opens a school door, closes a prison.”*
>
> – Victor Hugo

CH *N* – SOLVING QUADRATICS BY FACTORING

We're pretty good by now at solving equations like 2(3x – 4) + 8 = –10(x + 1), and you've probably had your fill of word problems which can be solved by such equations. But some of the most important applications in algebra, science, and business involve equations where the variable is *squared*. For instance, the following is called a ***quadratic equation***, due simply to the fact that the variable is being squared:



$$x^2 - 10x + 16 = 0$$

Now that's an equation!

☐ *CHECKING THE SOLUTIONS OF A QUADRATIC EQUATION*

First, let's verify that $\underline{x = 2}$ is a solution of the above equation (don't worry about where the 2 came from):

$$x^2 - 10x + 16 = 0$$

$$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \overset{?}{=} 0$$

$$4 - 20 + 16 \overset{?}{=} 0$$

$$-16 + 16 \overset{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

Fine — we have a solution. Here comes the (possibly) surprising fact: This equation has another solution, namely <u>$x = 8$</u>. Watch this:

$$x^2 - 10x + 16 = 0$$

$$\mathbf{8}^2 - 10(\mathbf{8}) + 16 \overset{?}{=} 0$$

$$64 - 80 + 16 \overset{?}{=} 0$$

$$-16 + 16 \overset{?}{=} 0$$

$$0 = 0 \quad ✓$$

One equation with <u>two</u> solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: We call it a ***quadratic equation***.

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

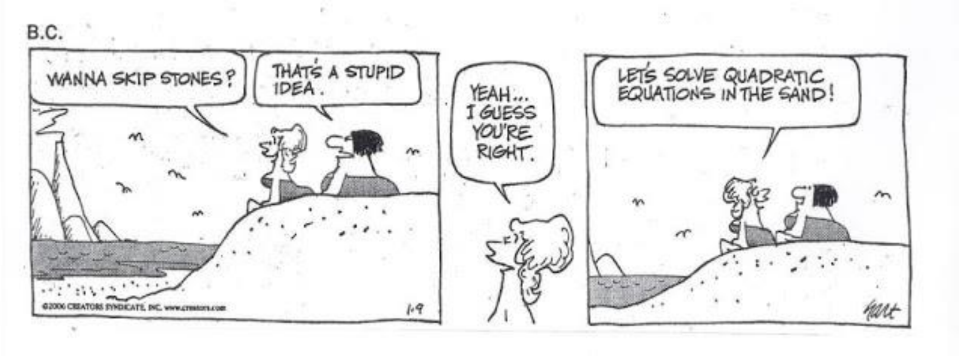
   a. $x^2 + 3x - 10 = 0$ $\quad x = -5;\ x = 2$

   b. $n^2 - 25 = 0$ $\quad n = 5;\ n = -5$

   c. $a^2 + 7a = -12$ $\quad a = -3;\ a = -4$

   d. $w^2 = 7w + 18$ $\quad w = 9;\ w = -2$

   e. $2y^2 + 8y = 0$ $\quad y = -4;\ y = 0$



☐ *QUADRATIC EQUATIONS*

Recall the equation from part d. of the homework:

$$w^2 = 7w + 18$$

Because the variable is squared, this is an example of a ***quadratic equation*** (from "quadrus," Latin for a 4-sided *square*). We can write this quadratic equation in **standard form** by subtracting $7w$ from each side of the equation and then subtracting 18 from each side of the equation:

$$w^2 - 7w - 18 = 0 \quad \longleftarrow \text{ A quadratic equation in } \textbf{\textit{standard form}}$$

Notice that the quadratic (squared) term is written first, the linear term (the one with the variable to the 1st power) comes next, and the constant term comes last; the right side of the equation is 0. We also note that the coefficient of the squared term is 1, the coefficient of the linear term is –7, and the constant term is –18.

The ***coefficient*** tells us how many of a particular thing we have. If we have 9*n*, then we have 9 *n*'s, and so the coefficient of 9*n* is 9. The coefficient of the term –7*xy* is –7, and the coefficient of $13w^2$ is 13.

Sometimes the coefficient is "implied." For example, the coefficient of u^2 is 1 (since $u^2 = 1u^2$), and the coefficient of –*m* is –1 (since –*m* = –1*m*).

We can solve some quadratic equations in our head. For instance, consider the quadratic equation

$$x^2 = 25$$

What number, when squared, equals 25? Recalling the homework above, we're not surprised to learn that there are two solutions to this equation: 5 and –5. After all, $5^2 = 25$, and $(-5)^2 = 25$. We conclude that the solutions of the quadratic equation $x^2 = 25$ are $x = \mathbf{5}$ and $x = \mathbf{-5}$. So, as we study this and future chapters, keep in mind that a quadratic equation is likely (but not necessarily) to have two different solutions.

❑ SOLVING QUADRATIC EQUATIONS BY FACTORING

Suppose we came across the statement

$$AB = 0$$

This says that the product of A and B is 0. Can we deduce anything? The only way this can occur is if either $A = 0$ **or** $B = 0$. This is the basis of the method of this chapter.

So, presented, for instance, with the quadratic equation

$$2w^2 - 31w + 84 = 0$$

we use the same logic as above, but we have to first convert the left side of the equation into a multiplication problem (like AB). And to convert an expression with three terms (containing addition and subtraction) into a multiplication problem, we *FACTOR!*

> If $AB = 0$
> then $A = 0$ or $B = 0$.

Our process in solving the equation is to **factor** the quadratic expression on the left side of the equation (noting that the right side is 0), set each factor to 0, and then solve each linear equation.

EXAMPLE 1: **Solve the quadratic equation $2w^2 - 31w + 84 = 0$.**

Solution:

$$\begin{array}{lll} & 2w^2 - 31w + 84 = 0 & \text{(the original equation)} \\ \Rightarrow & (2w - 7)(w - 12) = 0 & \text{(factor the quadratic)} \\ \Rightarrow & 2w - 7 = 0 \quad \text{or} \quad w - 12 = 0 & \text{(set each factor to 0)} \\ \Rightarrow & 2w = 7 \quad \text{or} \quad w = 12 & \text{(solve each equation)} \\ \Rightarrow & w = \dfrac{7}{2} \quad \text{or} \quad w = 12 & \end{array}$$

Thus, the solutions for w are

$$\boxed{\frac{7}{2},\ 12}$$

EXAMPLE 2: **Solve for x: $3x^2 - 7x - 40 = 0$**

Solution: This equation is in standard quadratic form, so it's all set to factor:

$$
\begin{array}{lll}
 & 3x^2 - 7x - 40 = 0 & \text{(the original equation)} \\
\Rightarrow & (3x+8)(x-5) = 0 & \text{(factor the left side)} \\
\Rightarrow & 3x + 8 = 0 \quad \text{or} \quad x - 5 = 0 & \text{(set each factor to 0)} \\
\Rightarrow & x = \dfrac{-8}{3} = -\dfrac{8}{3} \quad \text{or} \quad x = 5 & \text{(solve each equation)}
\end{array}
$$

Thus, the final solutions to the quadratic equation are $\boxed{5,\ -\frac{8}{3}}$

It's time to practice our equation-checking. Letting $x = 5$ in the original equation $3x^2 - 7x - 40 = 0$ gives

$$3(\mathbf{5})^2 - 7(\mathbf{5}) - 40 = 3(25) - 7(5) - 40 = 75 - 35 - 40 = 0 \quad ✓$$

Trying $x = -\frac{8}{3}$ in the original equation produces

$$3\left(-\frac{\mathbf{8}}{\mathbf{3}}\right)^2 - 7\left(-\frac{\mathbf{8}}{\mathbf{3}}\right) - 40 = 3\left(\frac{64}{9}\right) - 7\left(-\frac{8}{3}\right) - 40$$

$$= \frac{64}{3} + \frac{56}{3} - \frac{120}{3} = \frac{120}{3} - \frac{120}{3} = 0 \quad ✓$$

EXAMPLE 3: **Solve for y: $9y^2 - 16 = 0$**

Solution: It's a good-looking quadratic (even though the middle term, the y-term, is missing), so let's factor and set the factors to zero:

$$
\begin{array}{lll}
 & 9y^2 - 16 = 0 & \text{(the original equation)} \\
\Rightarrow & (3y+4)(3y-4) = 0 & \text{(factor the left side)}
\end{array}
$$

$\Rightarrow \quad 3y + 4 = 0 \quad \text{or} \quad 3y - 4 = 0$ (set each factor to 0)

$\Rightarrow \quad 3y = -4 \quad \text{or} \quad 3y = 4$ (remove the 4's)

$\Rightarrow \quad y = -\frac{4}{3} \quad \text{or} \quad y = \frac{4}{3}$ (divide both equations by 3)

Therefore, the solutions of the equation are

$\boxed{\frac{4}{3}, -\frac{4}{3}}$ which can also be written $\pm\frac{4}{3}$.

EXAMPLE 4: **Solve for u: $9u^2 = 42u - 49$**

Solution: This quadratic equation is not in standard form, so the first two steps will be to transform it into standard form:

$9u^2 = 42u - 49$ (the original equation)

$\Rightarrow \quad 9u^2 - 42u = -49$ (subtract $42u$)

$\Rightarrow \quad 9u^2 - 42u + 49 = 0$ (add 49 — standard form)

$\Rightarrow \quad (3u - 7)(3u - 7) = 0$ (factor)

$\Rightarrow \quad 3u - 7 = 0 \quad \text{or} \quad 3u - 7 = 0$ (set each factor to 0)

$\Rightarrow \quad u = \frac{7}{3} \quad \text{or} \quad u = \frac{7}{3}$ (solve each equation)

We obtained two solutions, but they're equal to each other, so there's just one solution (or are there two solutions which are the same?):

$$\boxed{\frac{7}{3}}$$

Homework

2. Solve each quadratic equation by factoring:

a. $w^2 - 12w + 35 = 0$

b. $2x^2 - 13x + 15 = 0$

c. $n^2 - 25n + 150 = 0$

d. $6a^2 - 31a + 40 = 0$

e. $x^2 + 2x - 15 = 0$

f. $y^2 - 14y + 49 = 0$

g. $z^2 - 9 = 0$

h. $3h^2 - 17h + 10 = 0$

i. $16u^2 - 8u + 1 = 0$

j. $25w^2 - 4 = 0$

3. Solve each quadratic equation by factoring:

a. $2x^2 + 10x - 28 = 0$

b. $25y^2 = 4$

c. $2z^2 + 2 = -4z$

d. $4a^2 = 3 - 4a$

e. $6u^2 = 47u + 8$

f. $0 = 4t^2 + 8t + 3$

g. $x^2 + x - 42 = 0$

h. $2n^2 + n - 3 = 0$

i. $4x^2 = 1$

j. $a^2 + 12a + 36 = 0$

k. $9k^2 = 9k - 2$

l. $9n^2 - 30n + 25 = 0$

m. $6z^2 = 10z$

n. $49t^2 = 4$

o. $25h^2 = 30h - 9$

p. $6x^2 - 7x - 10 = 0$

q. $0 = 6y^2 + y - 2$

r. $4n^2 = 25n + 21$

s. $16q^2 - 24q + 9 = 0$

t. $64t^2 - 9 = 0$

u. $-n^2 + n + 56 = 0$ [Hint: multiply each side by –1]

v. $-2x^2 + x + 3 = 0$

w. $-14a^2 = 3 - 13a$

☐ *FINDING A QUADRATIC EQUATION FROM THE SOLUTIONS*

When solving a quadratic equation by factoring, we set the quadratic to zero, factor the quadratic, set each of the two linear factors to zero, and then solve the two linear equations. However, for the next example, we have to find an equation given that we know its solutions; to do this we simply reverse the process of solving the equation.

EXAMPLE 5: **Find a quadratic equation whose solutions are 3 and –12.**

Solution:

You can use any variable you like, but I'll use x:

$x = 3$ OR $x = -12$ (the given two solutions)

$\Rightarrow$ $x - 3 = 0$ OR $x + 12 = 0$ (rewrite so that each equation is 0)

$\Rightarrow$ $(x - 3)(x + 12) = 0$ (the product of the factors is 0)

$\Rightarrow$ $\boxed{x^2 + 9x - 36 = 0}$ (double distribute)

Note: The above example asked for "a" quadratic equation, not "the" quadratic equation. Why? Because there are an infinite number of quadratic equations which have 3 and –12 as their solutions.

For example, check out $10x^2 + 90x - 360 = 0$. If you solve it by the factoring method, you'd likely would start by factoring out the GCF of 10:

$$10x^2 + 90x - 360 = 0$$

$$\Rightarrow \quad \mathbf{10}(x^2 + 9x - 36) = 0$$

$$\Rightarrow \quad 10(x - 3)(x + 12) = 0.$$

Dividing each side of the equation by 10 gives the equation

$$(x - 3)(x + 12) = 0,$$

whose solutions are exactly those required by the problem, 3 and -12. Therefore, a perfectly fine answer to the question would be the quadratic equation $10x^2 + 90x - 360 = 0$. In fact, for any number $\boldsymbol{k} \neq 0$, the quadratic equation $\boldsymbol{k}(x^2 + 9x - 36) = 0$ would properly answer the question. Nevertheless, let's agree that the best answer is the simplest one.

Homework

Find a quadratic equation (in standard form) with the given solutions:

4. 2 and -7
5. $\frac{2}{3}$ and 1
6. $-\frac{1}{2}$ and $\frac{4}{5}$
7. 0 and 9
8. The only solution is 10
9. The only solution is $-\frac{2}{3}$

Review Problems

10. Solve by factoring: $25y^2 = 1$

11. Solve by factoring: $n^2 + 16n + 64 = 0$

12. Solve by factoring: $26y^2 = 5y$

13. Solve by factoring: $25t^2 + 30t + 9 = 0$

14. Solve by factoring: $24x^2 = 14x + 3$

15. Solve by factoring: $81h^2 = 25$

16. Find a quadratic equation (in standard form) with the given solutions:

a. -3 and -5

b. $\frac{3}{2}$ and -2

c. $-\frac{3}{5}$ and $\frac{1}{2}$

d. 0 and $\frac{5}{6}$

e. The only solution is -8

f. The only solution is $\frac{7}{4}$

❑ *To ∞ and Beyond*

A. Find a quadratic equation (in standard form) with the given solutions:

1. $\pm\sqrt{7}$

2. $1+\sqrt{2}$ and $1-\sqrt{2}$

3. $-1+\sqrt{5}$ and $-1-\sqrt{5}$

4. 1 and 2 and 3

B. Solve for x: $x^3 + 3x^2 - 10x = 0$

Hint: There are three solutions.

C. Solve for n: $n^4 - 25n^2 + 144 = 0$

Hint: There are four solutions.

D. Solve for x: $\dfrac{x^2 - 5x + 6}{\sqrt[7]{\sin^2 x - \ln x + x^2 - \csc^3(e^x)}} = 0$

Solutions

1. Just plug in the proposed solutions and make sure they work.

2. a. 5, 7 b. $5, \frac{3}{2}$ c. 10, 15 d. $\frac{8}{3}, \frac{5}{2}$

e. 3, –5 f. 7 g. ±3 h. $5, \frac{2}{3}$

i. $\frac{1}{4}$ j. $\pm\frac{2}{5}$

3. a. 2, –7 b. $\pm\frac{2}{5}$ c. –1 d. $\frac{1}{2}, -\frac{3}{2}$

e. $8, -\frac{1}{6}$ f. $-\frac{1}{2}, -\frac{3}{2}$ g. 6, –7 h. $1, -\frac{3}{2}$

i. $\pm\frac{1}{2}$ j. –6 k. $\frac{1}{3}, \frac{2}{3}$ l. $\frac{5}{3}$

m. $0, \frac{5}{3}$ n. $\pm\frac{2}{7}$ o. $\frac{3}{5}$ p. $2, -\frac{5}{6}$

q. $\frac{1}{2}, -\frac{2}{3}$ r. $7, -\frac{3}{4}$ s. $\frac{3}{4}$ t. $\pm\frac{3}{8}$

u. 8, –7 v. $-1, \frac{3}{2}$ w. $\frac{3}{7}, \frac{1}{2}$

4. $x = 2 \text{ OR } x = -7 \Rightarrow x - 2 = 0 \text{ OR } x + 7 = 0 \Rightarrow (x-2)(x+7) = 0$
$\Rightarrow x^2 + 5x - 14 = 0$

5. $x = \frac{2}{3} \text{ OR } x = -1 \Rightarrow x - \frac{2}{3} = 0 \text{ OR } x + 1 = 0 \Rightarrow \left(x - \frac{2}{3}\right)(x+1) = 0$
$\Rightarrow x^2 + x - \frac{2}{3}x - \frac{2}{3} = 0 \Rightarrow x^2 + \frac{1}{3}x - \frac{2}{3} = 0 \Rightarrow 3x^2 + x - 2 = 0$

6. $10x^2 - 3x - 4 = 0$

7. $x^2 - 9x = 0$

8. $(x - 10)(x - 10) = 0 \Rightarrow x^2 - 20x + 100 = 0$

9. $9x^2 + 12x + 4 = 0$

10. $\pm\frac{1}{5}$ **11**. -8 **12**. $0, \frac{5}{26}$

13. $-\frac{3}{5}$ **14**. $\frac{3}{4}, -\frac{1}{6}$ **15**. $\pm\frac{5}{9}$

16. a. $x^2 + 8x + 15 = 0$ b. $2x^2 + x - 6 = 0$

c. $10x^2 + x - 3 = 0$ d. $6x^2 - 5x = 0$

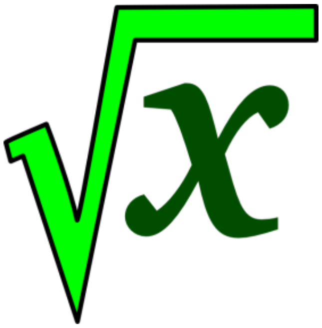
e. $x^2 + 16x + 64 = 0$ f. $16x^2 - 56x + 49 = 0$

"Every job is a self-portrait
of the person who did it.
Autograph your work with excellence."

– Unknown

CH XX – SOLVING QUADRATICS BY TAKING SQUARE ROOTS

❑ *INTRODUCTION*



Let's look at the quadratic equation

$$x^2 = 100$$

One way you may have learned to solve this equation is by **factoring**:

$$x^2 = 100 \;\Rightarrow\; x^2 - 100 = 0 \;\Rightarrow\; (x + 10)(x - 10) = 0$$

$$\Rightarrow \quad x + 10 = 0 \text{ or } x - 10 = 0 \;\Rightarrow\; x = -10 \text{ or } x = 10$$

The solutions of the quadratic equation $x^2 = 100$ are simply $x = \boldsymbol{\pm 10}$.

But there's a second way to solve a quadratic equation like $x^2 = 100$: Just take the square root of each side of the equation, <u>remembering that the number 100 has **TWO** square roots</u>, namely 10 and –10. Therefore, $x = \boldsymbol{\pm 10}$, and no factoring is required for this simple quadratic equation.

For another example, let's solve the quadratic equation $n^2 = 30$. Remembering that 30 has <u>two</u> square roots, we calculate n to be $\boldsymbol{\pm\sqrt{30}}$.

For a third example, where we will need to simplify the radical, consider the quadratic equation $y^2 = 72$. When we take the square root of each side of the equation — *and when we remember that* 72 *has* <u>*two*</u> *square roots* — we see that

$$y^2 = 72 \;\Rightarrow\; y = \pm\sqrt{72} = \pm\sqrt{36\cdot 2} = \pm\sqrt{36}\cdot\sqrt{2} = \pm 6\sqrt{2}$$

In short, the solutions of $y^2 = 72$ are $\boldsymbol{\pm 6\sqrt{2}}$.

☐ THE SQUARE ROOT THEOREM

Now for the general statement:

The solutions of the equation

$x^2 = A$ are $x = \pm\sqrt{A}$

The
Square Root
Theorem

Notes:

1) The value of A in the Square Root Theorem is assumed to be zero or positive; that is, $A \geq 0$. Otherwise, the square root will be imaginary, which may or may not represent a solution in your Algebra course; in this chapter, the square root of a negative number will not count as a solution.

2) How do students usually mess up this kind of equation? By forgetting to include both square roots (that is, they forget the "±" sign). DON'T MESS UP!

EXAMPLE 1: **Solve the quadratic equation: $(x + 7)^2 = 81$**

Solution: According to the Square Root Theorem, we can remove the squaring by taking the square root of both sides of the equation, remembering that the number 81 has two square roots:

$(x + 7)^2 = 81$ (the original equation)

$\Rightarrow \quad x + 7 = \pm\sqrt{81}$ (the Square Root Theorem)

$$\Rightarrow \quad x + 7 = \pm 9 \qquad (\sqrt{81} = 9)$$

$$\Rightarrow \quad x = -7 \pm 9 \qquad \text{(subtract 7 from each side)}$$

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$\boxed{x = 2 \text{ or } -16}$$

EXAMPLE 2: **Solve for y: $(y - 3)^2 = 32$**

Solution: First we need to remove, or undo, the squaring in this quadratic equation. This is where we apply the Square Root Theorem:

$$y - 3 = \pm\sqrt{32} \qquad \text{(32 has \underline{two} square roots)}$$

To isolate the y, add 3 to both sides:

$$y = 3 \pm \sqrt{32}$$

Since $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$, we simplify our solution:

$$\boxed{y = 3 \pm 4\sqrt{2}}$$

Be sure it's clear to you that we have written two solutions to our quadratic equation:

$$3 + 4\sqrt{2} \text{ and } 3 - 4\sqrt{2}$$

EXAMPLE 3: **Solve for n: $(n - 3)^2 = -49$**

Solution: Applying the Square Root Theorem to remove the squaring gives us the equation

$$n - 3 = \pm\sqrt{-49}$$

We needn't go any further; after all, we're not dealing with imaginary numbers in this chapter. So we're done right here, and we conclude that the equation has

$$\boxed{\text{No Solution}}$$

Homework

1. Solve each equation by applying the Square Root Theorem:

a. $x^2 = 144$

b. $y^2 = 51$

c. $z^2 = 72$

d. $a^2 = 0$

e. $b^2 = -9$

f. $(x + 1)^2 = 25$

g. $(n - 3)^2 = 100$

h. $(u + 10)^2 = 1$

i. $(a - 5)^2 = 32$

j. $(b + 7)^2 = 50$

k. $(w + 13)^2 = -4$

l. $(m - 3)^2 = 75$

☐ *SPLITTING RADICALS IN DIVISION*

Do you remember the rule about splitting the square root of a product: $\sqrt{ab} = \sqrt{a}\sqrt{b}$? The same kind of rule works for division. For example, we know that $\sqrt{\frac{9}{25}} = \frac{3}{5}$, since $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Now let's work it out by "splitting" the radical:

$$\underbrace{\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}}}_{\text{split the radical}} = \frac{3}{5}\text{, the same answer!}$$

In short, $\sqrt{\boldsymbol{\frac{a}{b}}} \boldsymbol{=} \boldsymbol{\frac{\sqrt{a}}{\sqrt{b}}}$ [assuming that $a \geq 0$ and $b > 0$]

Homework

2. Calculate each square root by "splitting" the radical:

a. $\sqrt{\frac{1}{25}}$ b. $\sqrt{\frac{9}{49}}$ c. $\sqrt{\frac{16}{81}}$ d. $\sqrt{\frac{100}{121}}$ e. $\sqrt{\frac{144}{36}}$

We conclude this chapter with an example that requires us to "split" the square root of a fraction.

<u>EXAMPLE 4:</u> **Solve for z:** $\left(z-\frac{4}{5}\right)^2 = \frac{8}{25}$

<u>Solution:</u> Using the Square Root Theorem, we take the square root of each side of the equation, remembering that $\frac{8}{25}$ has <u>two</u> square roots:

$$z-\frac{4}{5} = \pm\sqrt{\frac{8}{25}} \qquad \text{(the Square Root Theorem)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \sqrt{\frac{8}{25}} \qquad \text{(isolate the } z \text{ by adding 4/5)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{\sqrt{8}}{\sqrt{25}} \qquad \text{(split the radical)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{2\sqrt{2}}{5} \qquad \text{(simplify both square roots)}$$

$$\Rightarrow \quad \boxed{z = \frac{4 \pm 2\sqrt{2}}{5}} \qquad \text{(combine into a single fraction)}$$

Again, note that we have found <u>two</u> solutions. They may be ugly, but both of them satisfy the equation.

Homework

3. Solve each equation by applying the Square Root Theorem:

a. $\left(x-\frac{1}{2}\right)^2 = \frac{3}{4}$ b. $\left(t+\frac{2}{3}\right)^2 = \frac{1}{9}$ c. $\left(z-\frac{4}{5}\right)^2 = \frac{19}{25}$

d. $\left(x+\frac{3}{5}\right)^2 = \frac{12}{25}$ e. $\left(b-\frac{9}{10}\right)^2 = \frac{81}{100}$ f. $\left(g-\frac{3}{7}\right)^2 = \frac{24}{49}$

Review Problems

4. Solve each equation by applying The Square Root Theorem:

a. $x^2 = 121$ b. $y^2 = 50$ c. $z^2 = 0$

d. $n^2 = -25$ e. $t^2 = 288$ f. $a^2 - 14 = 0$

g. $(x+1)^2 = 75$ h. $(c-3)^2 = 10$ i. $(x+10)^2 = -1$

j. $\left(w+\frac{1}{2}\right)^2 = \frac{3}{4}$ k. $\left(u-\frac{4}{3}\right)^2 = \frac{26}{9}$ l. $\left(a+\frac{8}{5}\right)^2 = \frac{32}{25}$

Solutions

1. a. $x = \pm 12$ b. $y = \pm\sqrt{51}$ c. $z = \pm 6\sqrt{2}$

d. $a = 0$ e. No solution f. $x = 4, -6$

g. $n = 13, -7$ h. $u = -9, -11$ i. $a = 5 \pm 4\sqrt{2}$

j. $b = -7 \pm 5\sqrt{2}$ k. No solution l. $m = 3 \pm 5\sqrt{3}$

2. a. $\sqrt{\frac{1}{25}} = \frac{\sqrt{1}}{\sqrt{25}} = \frac{1}{5}$ b. $\frac{3}{7}$ c. $\frac{4}{9}$ d. $\frac{10}{11}$ e. 2

3. a. $\left(x - \frac{1}{2}\right)^2 = \frac{3}{4} \Rightarrow x - \frac{1}{2} = \pm\sqrt{\frac{3}{4}} \Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{3}}{\sqrt{4}} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = \frac{1 \pm \sqrt{3}}{2}$

b. $t = -\frac{1}{3}, -1$ c. $z = \frac{4 \pm \sqrt{19}}{5}$ d. $x = \frac{-3 \pm 2\sqrt{3}}{5}$

e. $b = \frac{9}{5}, 0$ f. $g = \frac{3 \pm 2\sqrt{6}}{7}$

4. a. $x = \pm 11$ b. $y = \pm 5\sqrt{2}$ c. $z = 0$

d. No solution e. $t = \pm 12\sqrt{2}$ f. $a = \pm\sqrt{14}$

g. $x = -1 \pm 5\sqrt{3}$ h. $c = 3 \pm \sqrt{10}$ i. No solution

j. $w = \frac{-1 \pm \sqrt{3}}{2}$ k. $u = \frac{4 \pm \sqrt{26}}{3}$ l. $a = \frac{-8 \pm 4\sqrt{2}}{5}$

"Learning is
a treasure
that will follow
its owner
everywhere."

Chinese Proverb

CH NN – DERIVING THE QUADRATIC FORMULA

This chapter assumes that you have finished solving quadratic equations by *Completing the Square*. You might understand at this point in your studies that the *factoring* method for solving quadratic equations seldom works, while Completing the Square *always* works.

And it works beautifully whether the solutions are integers, fractions (rational numbers), or radicals (irrational numbers). It even tells us when an equation has no solutions in ℝ, the system of real numbers (which occurs when we reach the square root of a negative number).

But Completing the Square is quite boring and not the easiest thing in the world to do. There must be some way to complete the square on a "generic" quadratic equation, $ax^2 + bx + c = 0$, and end up with a formula that can be used to solve any quadratic equation more efficiently, whether it's factorable or not.

"I FOUND OUT TODAY IN SCHOOL THAT I'M QUADRATIC EQUATION CHALLENGED!"

☐ *THE QUADRATIC FORMULA*

We will start with a generic quadratic equation in standard form, and then try to *complete the square* . . . here we go:

$$ax^2 + bx + c = 0$$

[We assume $a > 0$, because if it's not, we can multiply through by -1 and make $a > 0$.]

The technique of completing the square requires that we always have a leading coefficient of 1, and so the first step is to divide each side of the equation by a (which we know is not zero — otherwise, we wouldn't have a quadratic equation in the first place):

Divide each side of the equation by a:

$$\frac{ax^2+bx+c}{a} = \frac{0}{a}$$

To simplify the left side of the equation, just be sure you divide <u>each</u> term by a:

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

(basically the reverse of adding fractions)

And then simplify each side:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

(since $\frac{bx}{a} = \frac{b}{a} \cdot \frac{x}{1} = \frac{b}{a} \cdot x = \frac{b}{a}x$)

Now bring the constant to the right side of the equation; that is, subtract $\frac{c}{a}$ from each side of the equation (we could have done this step first):

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

It's now time to compute the "magic number," the quantity that will "complete the square." We start with the coefficient of the linear term, in this case $\frac{b}{a}$, take half of it, and then square that result:

$$\frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}, \text{ and then } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \boxed{\boldsymbol{\frac{b^2}{4a^2}}} \longleftarrow$$

The magic number

It's time to add the magic number to each side of the equation:

$$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = -\frac{c}{a} + \boxed{\frac{b^2}{4a^2}}$$

There are two things to do now: factor on the left and combine the fractions on the right. Let's factor first:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Notice that the variable x occurs just once in the entire equation (a good thing), buried amidst the parentheses on the left side of the equation.

Now we'll combine the fractions on the right side of the equation into a single fraction. Since the LCD of the denominators is $4a^2$, we need to multiply the top and the bottom of the first fraction by $4a$:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\left[\frac{\mathbf{4a}}{\mathbf{4a}}\right] + \frac{b^2}{4a^2} \qquad \left(\text{since } \frac{4a}{4a} = 1\right)$$

or, $$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \text{(We now have a common denominator)}$$

Adding the fractions yields the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Using the commutative property for addition, we can reverse the two terms in the numerator of the right side of the equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The next step we need in order to isolate the x is to take the square root of each side of the equation, remembering that every non-negative number has <u>two</u> square roots, denoted $\pm\sqrt{}$:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{(2 square roots!)}$$

To isolate the x (which is the whole point of this endeavor), we'll bring the term $\frac{b}{2a}$ to the right side of the equation (and put it in front):

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

We've isolated the x. Yay! The rest is simplifying.

Now it's appropriate to split the radical into two separate radicals:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

Recall: $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ (for $y > 0$)

Simplify the radical in the denominator:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Technically speaking, $\sqrt{4a^2} = 2|a|$. But since we assumed at the outset that $a > 0$, the radical becomes $2a$.

Finally, we combine the two fractions into a single fraction (can you believe the two fractions already have the same denominator? ☺):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In a single sentence: We've isolated the x in the quadratic equation $ax^2 + bx + c = 0$ by *completing the square*!

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by **The Quadratic Formula**:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

❑ NOTES ON THE QUADRATIC FORMULA

- The Quadratic Formula can be used to solve any quadratic equation, whether or not it's factorable, and whether its solutions are integers (whole numbers and their opposites), fractions, or even radicals (like $\sqrt{2}$). And if the quadratic equation has no solutions at all in the real numbers (that is, they have imaginary solutions), the Quadratic Formula will tell us that, too.

- Because of the ± sign, there are potentially two solutions. In fact, the two solutions of the quadratic equation $ax^2 + bx + c = 0$ can be written separately if you prefer:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

- The Quadratic Formula is a single fraction:

It is **NOT** $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

If you don't write it as a single fraction at every stage of your solution, either you or your instructor will misread it and you'll be the one to lose points on the problem.

Homework

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?

2. How many solutions of the quadratic equation will there be if the quantity inside the square root sign (called the *radicand*) is 100?

3. How many solutions of the quadratic equation will there be if the radicand is 30?

4. How many solutions of the quadratic equation will there be if the radicand is 0?

5. How many solutions of the quadratic equation will there be if the radicand is –9?

6. Consider the possibility that the value of a is **0** in the quadratic equation $ax^2 + bx + c = 0$. Answer the question, Does the Quadratic Formula apply, in two ways:

   a. Consider what happens to the quadratic equation if $a = \mathbf{0}$.

   b. Now analyze what happens if we actually allow $a = \mathbf{0}$ to wriggle its way into the Quadratic Formula.

7. Consider the quantity "$\boldsymbol{b^2 - 4ac}$," the radicand inside the square root sign of the Quadratic Formula.

   a. If $b^2 - 4ac > 0$, then the equation has ____ solution(s).

   b. If $b^2 - 4ac = 0$, then the equation has ____ solution(s).

   c. If $b^2 - 4ac < 0$, then the equation has ____ solution(s).

   [The quantity $b^2 - 4ac$ is called the ***discriminant*** of the quadratic polynomial $ax^2 + bx + c$.]

Solutions

1. The plus/minus sign: ±

2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front of the radical, there will be <u>two</u> solutions.

3. Even though $\sqrt{30}$ doesn't result in a whole number, there's still a plus/minus sign in front of it, so there will also be <u>two</u> solutions.

4. Since $\sqrt{0} = 0$, and since ±0 is just the single number 0, the plus/minus sign basically disappears, leaving <u>one</u> solution.

5. Since $\sqrt{-9}$ does not exist in early algebra classes, the Quadratic Formula has no meaning, and so there are <u>no</u> solutions.

6. First, if $a = 0$, the quadratic equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned months ago how to solve this equation:

$$bx + c = 0 \Rightarrow bx = -c \Rightarrow x = -\frac{c}{b}$$

Second, if we ignore the above fact and use $a = 0$ in the Quadratic Formula, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which is undefined. However you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

7. a. 2 b. 1 c. 0 (assuming we want to stay within the real numbers.)

Postscript:

The derivation of the Quadratic Formula in this chapter was the classic method, using the method of completing the square. This was a fine technique to use, since completing the square will be used in your later studies of circles, parabolas, and other shapes — and the more practice, the better.

But there's another way to derive the Quadratic Formula, perhaps a way that may be easier for you to follow. Let's do it now — beginning, of course, with our quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

First, multiply each side of the equation by $4a$:

$$\mathbf{4a}(ax^2 + bx + c) \;=\; \mathbf{4a}(0)$$

Distribute on the left and simplify on the right:

$$4a^2x^2 + 4abx + 4ac \;=\; 0$$

Bring the $4ac$ to the right side:

$$4a^2x^2 + 4abx \;=\; -4ac$$

Next, add b^2 to each side of the equation:

$$4a^2x^2 + 4abx + \boxed{b^2} \;=\; -4ac + \boxed{b^2}$$

Now factor the left side and flip the terms on the right side:

$$(2ax + b)^2 \;=\; b^2 - 4ac$$

Then remove the squaring by taking both square roots on the right:

$$2ax + b \;=\; \pm\sqrt{b^2 - 4ac}$$

Subtract b from each side of the equation:

$$2ax \;=\; -b \pm \sqrt{b^2 - 4ac}$$

And last, divide each side of the equation by $2a$:

$$\boxed{x \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

WE DID IT!!

CH XX –THE QUADRATIC FORMULA, IRRATIONAL SOLUTIONS

❑ INTRODUCTION

Your first experience with the Quadratic Formula probably involved solving a quadratic equation like $6x^2 + 31x + 40 = 0$, whose solutions are $-\frac{5}{2}$ and $-\frac{8}{3}$. In fact, all of the answers back then were either positive or negative whole numbers, or fractions, or zero. [In other words, all the solutions were *rational numbers*.]

You may have realized that we obtained these kinds of answers because the square-root part of the Quadratic Formula $\left(\sqrt{b^2 - 4ac}\right)$ always worked out to the square root of a perfect square, like $\sqrt{49}$ or $\sqrt{0}$.

But we know that there are square roots like $\sqrt{5}$. What would we do if we came up with a radical like that, assuming that no calculator is allowed? In this section, we'll see that there's really nothing special to worry about — just work the Quadratic Formula:

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 1: **Solve for x: $2x^2 - 3x - 1 = 0$**

Solution: For this quadratic equation, $a = 2$, $b = -3$, and $c = -1$. Inserting these values into the Quadratic Formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+8}}{4} = \boxed{\frac{3 \pm \sqrt{17}}{4}}$$

And that's it, because $\sqrt{17}$ cannot be simplified and the fraction cannot be reduced. If you'd like, the two solutions of the equation can be written separately:

$$x = \frac{3+\sqrt{17}}{4}, \frac{3-\sqrt{17}}{4}$$

Homework

1. Solve each quadratic equation:

a. $x^2 - 5x + 1 = 0$

b. $2n^2 + n - 7 = 0$

c. $3a^2 - 3a - 1 = 0$

d. $5t^2 + 7t + 1 = 0$

e. $5x^2 + 15x - 2 = 0$

f. $t^2 + t - 8 = 0$

g. $5h^2 - 13h = -5$

h. $7u^2 + u = 3$

i. $2k^2 + 7k = 8$

j. $2m^2 - 3 = -9m$

2. Solve for x: $-7x^2 + 5x + 1 = 0$

   Hint: Although not necessary, it's traditional to always make a, the leading coefficient, positive. This way, your solutions will more likely match those provided by your teacher or the textbook. Since $a = -7$ in this quadratic equation, we can multiply (or divide) each side of the equation by -1, yielding $7x^2 - 5x - 1 = 0$. Now you do the rest.

☐ *MORE QUADRATIC EQUATIONS*

EXAMPLE 2: **Solve for x: $2x^2 - 8x + 3 = 0$**

Solution: The values of a, b, and c for use in the Quadratic Formula are

$$a = 2 \qquad b = -8 \qquad c = 3$$

So now we state the Quadratic Formula, insert the three values, and then do a lot of arithmetic and simplifying until the final answer is exact (no decimals) and as clean as possible.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(The Quadratic Formula)}$$

$$= \frac{-(\mathbf{-8}) \pm \sqrt{(\mathbf{-8})^2 - 4(\mathbf{2})(\mathbf{3})}}{2(\mathbf{2})} \qquad (a = 2 \quad b = -8 \quad c = 3)$$

$$= \frac{8 \pm \sqrt{64 - 24}}{4} \qquad \text{(square and multiply)}$$

$$= \frac{8 \pm \sqrt{40}}{4} \qquad \text{(finish the arithmetic)}$$

$$= \frac{8 \pm \sqrt{4 \cdot 10}}{4} \qquad \text{(factor the radicand)}$$

$$= \frac{8 \pm 2\sqrt{10}}{4} \qquad \text{(simplify the radical)}$$

$$= \frac{\cancel{2}(4 \pm \sqrt{10})}{\cancel{2} \cdot 2} \qquad \text{(factor the numerator and reduce)}$$

We write our two solutions as

$$\boxed{x = \frac{4 \pm \sqrt{10}}{2}}$$

EXAMPLE 3: **Solve for y: $3y^2 = 13$**

Solution: We begin by subtracting 13 from each side of the equation to convert the equation into standard quadratic form:

$$3y^2 - 13 = 0$$

At this point, we see that $a = 3$, $b = 0$, and $c = -13$. Applying the Quadratic Formula:

$$y = \frac{-\mathbf{0} \pm \sqrt{\mathbf{0}^2 - 4(\mathbf{3})(\mathbf{-13})}}{2(\mathbf{3})} = \frac{\pm\sqrt{0+156}}{6} = \frac{\pm\sqrt{156}}{6}$$

$$= \frac{\pm\sqrt{4 \cdot 39}}{6} = \frac{\pm\sqrt{4}\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{2 \cdot 3} = \frac{\pm \cancel{2}\sqrt{39}}{\cancel{2} \cdot 3}$$

And we have our final answer (two of them!) in simplest form:

$$\boxed{y = \pm\frac{\sqrt{39}}{3}}$$

EXAMPLE 5: **Solve for n: $2n^2 = 3n - 4$**

Solution: If we convert to standard form, the equation becomes

$$2n^2 - 3n + 4 = 0$$

where $a = 2$, $b = -3$, and $c = 4$. The Quadratic Formula produces

$$n = \frac{-(\mathbf{-3}) \pm \sqrt{(\mathbf{-3})^2 - 4(\mathbf{2})(\mathbf{4})}}{2(\mathbf{2})} = \frac{3 \pm \sqrt{9-32}}{4} = \frac{3 \pm \sqrt{-23}}{4}$$

Remembering that $\sqrt{-23}$ is not a number in this Algebra course, we stop right here and declare that the given equation has

$$\boxed{\text{No Solution}}$$

Homework

3. Solve each quadratic equation:

a. $3x^2 - 6x - 1 = 0$
b. $2k^2 = 8k - 6$
c. $2n^2 - 10 = 0$
d. $y^2 = 49$
e. $14d^2 - 4d = 4$
f. $x^2 + 3x + 10 = 0$
g. $5a^2 + 7a - 2 = 0$
h. $u^2 = 40$
i. $3z^2 = 3z$
j. $5x^2 - x + 1 = 0$
k. $x(x - 3) = 7$
l. $y(y + 1) = y(y - 7) + 13$

4. Solve each quadratic equation:

a. $x^2 + 6x + 1 = 0$
b. $2y^2 + 7y = 2$
c. $-n^2 + 28 = 0$
d. $5d^2 = 7d + 1$
e. $x^2 = -4 - 5x$
f. $x^2 + 2x + 1 = 0$
g. $-3a^2 = 5 - 12a$
h. $2x^2 = 7$
i. $3z^2 = 12z$
j. $n^2 + 9 = 0$
k. $-4n^2 + 12n = 9$
l. $t^2 - 9 = 0$

Review Problems

5. Solve each quadratic equation:

a. $x^2 - x - 56 = 0$

b. $u^2 - 8u = 0$

c. $r^2 = 9$

d. $9k^2 + 49 = -42k$

e. $n^2 + 2n - 80 = 0$

f. $-3t^2 = 11t + 6$

g. $2m^2 - 7m = -6$

h. $8d^2 = 2$

i. $-3a^2 = 7a - 6$

j. $-8h^2 - 8h = 0$

k. $n^2 - 3n + 4 = 0$

l. $2p^2 - 3p - 5 = 0$

m. $-3m^2 - 8m + 3 = 0$

n. $-8z^2 + 2z = 0$

o. $-3w^2 + 3 = 0$

p. $5g^2 = 5g + 2$

q. $-3y^2 + 6y = -3$

r. $12t^2 - 8t - 16 = 0$

s. $3x^2 = -15x + 3$

t. $0 = 21g^2 + 9g - 15$

u. $-5z^2 + 4z = -7$

v. $8q^2 + 4q = 3$

w. $y^2 - 4y = -2$

x. $2x^2 - 2x - 3 = 0$

y. $0 = 14d^2 - 4d - 4$

z. $27x^2 - 6x - 15 = 0$

Solutions

1. a. $\frac{5 \pm \sqrt{21}}{2}$
b. $\frac{-1 \pm \sqrt{57}}{4}$
c. $\frac{3 \pm \sqrt{21}}{6}$
d. $\frac{-7 \pm \sqrt{29}}{10}$
e. $\frac{-15 \pm \sqrt{265}}{10}$
f. $\frac{-1 \pm \sqrt{33}}{2}$
g. $\frac{13 \pm \sqrt{69}}{10}$
h. $\frac{-1 \pm \sqrt{85}}{14}$
i. $\frac{-7 \pm \sqrt{113}}{4}$
j. $\frac{-9 \pm \sqrt{105}}{4}$

2. $\frac{5 \pm \sqrt{53}}{14}$

3. a. The quadratic equation $3x^2 - 6x - 1 = 0$ is in standard form. We can read the values $a = 3$, $b = -6$, and $c = -1$. Placing these values into the Quadratic Formula gives

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{16 \cdot 3}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6} = \frac{2(3 \pm 2\sqrt{3})}{2 \cdot 3} = \frac{\cancel{2}(3 \pm 2\sqrt{3})}{\cancel{2} \cdot 3} = \frac{3 \pm 2\sqrt{3}}{3}$$

b. 1, 3
c. $\pm\sqrt{5}$
d. ±7
e. $\frac{1 \pm \sqrt{15}}{7}$
f. No solution
g. $\frac{-7 \pm \sqrt{89}}{10}$
h. $\pm 2\sqrt{10}$
i. 0, 1
j. No solution
k. $\frac{3 \pm \sqrt{37}}{2}$
l. $\frac{13}{8}$

4. a. $-3 \pm 2\sqrt{2}$
b. $\frac{-7 \pm \sqrt{65}}{4}$
c. $\pm 2\sqrt{7}$
d. $\frac{7 \pm \sqrt{69}}{10}$
e. $-1, -4$
f. -1
g. $\frac{6 \pm \sqrt{21}}{3}$
h. $\pm\frac{\sqrt{14}}{2}$
i. 0, 4
j. No solution
k. $\frac{3}{2}$
l. ±3

5. a. $-7, 8$
b. $8, 0$
c. ± 3
d. $-\frac{7}{3}$
e. $-10, 8$
f. $-3, -\frac{2}{3}$
g. $2, \frac{3}{2}$
h. $\frac{1}{2}, -\frac{1}{2}$
i. $-3, \frac{2}{3}$
j. $-1, 0$
k. No solution
l. $\frac{5}{2}, -1$
m. $-3, \frac{1}{3}$
n. $0, \frac{1}{4}$
o. ± 1
p. $\frac{5 \pm \sqrt{65}}{10}$
q. $1 \pm \sqrt{2}$
r. $\frac{1 \pm \sqrt{13}}{3}$
s. $\frac{-5 \pm \sqrt{29}}{2}$
t. $\frac{-3 \pm \sqrt{149}}{14}$
u. $\frac{2 \pm \sqrt{39}}{5}$
v. $\frac{-1 \pm \sqrt{7}}{4}$
w. $2 \pm \sqrt{2}$
x. $\frac{1 \pm \sqrt{7}}{2}$
y. $\frac{1 \pm \sqrt{15}}{7}$
z. $\frac{1 \pm \sqrt{46}}{9}$

“Do, or do not.

There is no 'try'.”

– Yoda (*The Empire Strikes Back*)

CH XX – PREPARING FOR THE QUADRATIC FORMULA

As discussed in the introduction to *Preparing to Complete the Square*, the following is a fine, little, quadratic equation, which will NOT factor nicely:

$$x^2 + 7x + 5 = 0$$

❑ THE GENERAL QUADRATIC EQUATION

Consider a general quadratic equation in standard form:

$$\boxed{ax^2 + bx + c = 0}$$

where x represents the variable (the unknown) and a, b, and c are numbers. Notice that the squared term is first, the linear term is second, the constant term is third, and there's a 0 on the right side of the equation. The examples below show some quadratic equations along with their corresponding values of a, b, and c.

$3x^2 - 7x + 10 = 0$	⇨	$a = 3$	$b = -7$	$c = 10$
$y^2 - 9 = 0$	⇨	$a = 1$	$b = 0$	$c = -9$
$-4z^2 + 19z = 0$	⇨	$a = -4$	$b = 19$	$c = 0$
$2w^2 = 8 - 5w$	⇨	$a = 2$	$b = 5$	$c = -8$

To find the values of a, b, and c for this last equation, we need to rewrite the quadratic equation in standard form:

$2w^2 + 5w - 8 = 0$, from which the values of a, b, and c can be determined.

Homework

1. For each quadratic equation, first make sure that it's in standard form ($ax^2 + bx + c = 0$); then determine the values of a, b, and c:

a. $3x^2 + 9x + 17 = 0$

b. $2n^2 - 8n + 14 = 0$

c. $6y^2 + 2y - 2 = 0$

d. $5t^2 - 13t - 1 = 0$

e. $12a^2 + 13 = 0$

f. $x^2 - 13x = 0$

g. $u^2 - u + 1 = 0$

h. $-2w^2 - 19 = 0$

i. $-w^2 + 14w = 0$

j. $-z^2 - 99 = 0$

k. $2x^2 = 4x + 3$

l. $-3y^2 - 2y = -5$

m. $c^2 + 4 = 7c$

n. $6m^2 = 1 + m$

o. $18k^2 = 0$

p. $-3x^2 = -3x$

☐ *ORDER OF OPERATIONS*

Remember that exponents are done before multiplication, which itself comes before any addition or subtraction. For example,

$$\begin{aligned} & 12^2 - 4(2)(3) && \\ = \quad & 144 - 4(2)(3) && \text{(exponent first)} \\ = \quad & 144 - 24 && \text{(multiplication second)} \\ = \quad & \mathbf{120} && \text{(subtraction last)} \end{aligned}$$

For a second example,

$$
\begin{aligned}
& (-9)^2 - 4(3)(-5) & \\
= \quad & 81 - 4(3)(-5) & \text{(exponent first)} \\
= \quad & 81 - (-60) & \text{(multiplication second)} \\
= \quad & 81 + 60 & \text{(subtracting a negative is adding)} \\
= \quad & \mathbf{141} & \text{(addition last)}
\end{aligned}
$$

❑ Square Roots

$\sqrt{36} = 6$ $\qquad$ $-\sqrt{144} = -12$

$\sqrt{1} = 1$ $\qquad$ $\sqrt{0} = 0$

$\sqrt{225} = 15$ $\qquad$ $\sqrt{50} \approx 7.0711$

$\sqrt{-9}$ does not exist (in our algebra class) $\qquad$ *is approximately equal to*

❑ Opposites

Recall that the *opposite* of N is $-N$. So we note the following:

1) The opposite of a positive number is a negative number; for example, $-(+7) = -7$. Thus, if $b = 7$, then $-b = -7$.

2) The opposite of a negative number is a positive number; for example, $-(-12) = 12$. Thus, if $b = -12$, then $-b = 12$.

3) The opposite of a squared quantity (that isn't 0) is negative; for example, $-9^2 = -81$. On the other hand, don't forget that $(-9)^2 = 81$.

4) The opposite of 0 is 0.

❑ PLUS/MINUS

The symbol " ± " is read "***plus or minus***" and is just a short way of indicating two numbers at once. For example, if you want to indicate the two numbers 7 and –7, you may write just ±7. Another example might be $\pm\sqrt{81}$, which stands for the two numbers 9 and –9. But $\pm\sqrt{0}$ would only be 0, since +0 and –0 are really just two ways to express 0.

The equation $x^2 = 25$ has two solutions, namely that x could be 5 or –5, so we could write the solution as $x = \pm 5$.

Let's work out a specific problem containing the "plus or minus" sign.

Simplify: $\dfrac{10 \pm \sqrt{25}}{5}$

Taking the square root yields $\dfrac{10 \pm 5}{5}$.

Using the plus sign, we get $\dfrac{10+5}{5} = \dfrac{15}{5} = \mathbf{3}$,

while using the minus set yields $\dfrac{10-5}{5} = \dfrac{5}{5} = \mathbf{1}$.

Thus, the expression $\dfrac{10 \pm \sqrt{25}}{5}$ is merely a strange way to represent the numbers 3 and 1.

Homework

2. Evaluate each expression:

a. $13^2 - 4(3)(2)$

b. $0^2 - 4(2)(-1)$

c. $(-3)^2 - 4(-1)(-2)$

d. $(-5)^2 - 4(1)(0)$

e. $0^2 - 4(17)(0)$

f. $(-4)^2 - (-4)(-5)$

3. Evaluate each expression:

a. $\sqrt{49}$ b. $-\sqrt{6+3}$

c. $\sqrt{16}+\sqrt{9}$ d. $\sqrt{-25}$

e. $\sqrt{(-32)(-2)}$ f. $\sqrt{(-5)^2-4(1)(6)}$

g. $\sqrt{6^2-4(1)(9)}$ h. $\sqrt{(-10)^2-4(8)(2)}$

i. $\sqrt{1^2-4(1)(1)}$ j. $\sqrt{(-7)^2-4(4)(-2)}$

4. Evaluate each expression (the leading minus sign represents the *opposite* of the quantity which follows it):

a. $-(13)$ b. $-(-5)$ c. $-(-(-7))$

d. $-(+7)$ e. $-(-1)$ f. $-(0)$

5. What number or numbers are represented by each of the following?

a. ± 121 b. $\pm\sqrt{121}$ c. $\pm\sqrt{0}$ d. $\pm\sqrt{-1}$

6. Simplify each expression:

a. $100+\sqrt{49}$ b. $-5-\sqrt{121}$ c. $\frac{3+\sqrt{16}}{2}$ d. $\frac{-8-\sqrt{4}}{10}$

e. $7-\sqrt{49}$ f. $-10+\sqrt{100}$ g. $\frac{4\pm\sqrt{16}}{6}$ h. $\frac{-1\pm\sqrt{81}}{9}$

7. Evaluate the expression $b^2 - 4ac$ for the given values of a, b, and c:

a. $a=-10,\ b=-5,\ c=3$ b. $a=4,\ b=6,\ c=-1$

c. $a=5,\ b=0,\ c=7$ d. $a=-2,\ b=-3,\ c=0$

8. Evaluate the expression $\pm\sqrt{b^2-4ac}$ for the given values of a, b, and c:

a. $a = 2,\ b = -7,\ c = -15$ b. $a = 9,\ b = 30,\ c = 25$

c. $a = 1,\ b = 0,\ c = -81$ d. $a = 2,\ b = -3,\ c = 0$

☐ *THE ULTIMATE EXAMPLE*

For our final example of this chapter, let's evaluate the expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{for the values } a = 1,\ b = -5,\ c = 6$$

First change every variable to a pair of parentheses — this is not required, but it's a handy little trick to help us avoid errors:

$$= \frac{-(\ \ ) \pm \sqrt{(\ \ )^2 - 4(\ \ )(\ \ )}}{2(\ \ )}$$

$$= \frac{-(\mathbf{-5}) \pm \sqrt{(\mathbf{-5})^2 - 4(\mathbf{1})(\mathbf{6})}}{2(\mathbf{1})} \qquad \text{(substitute the given values)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{2} \qquad \text{(do exponents and multiplying)}$$

$$= \frac{5 \pm \sqrt{1}}{2} \qquad \text{(finish the inside of the radical)}$$

$$= \frac{5 \pm 1}{2} \qquad \text{(the positive square root of 1 is 1)}$$

$$= \frac{5+1}{2} \ \ or \ \ \frac{5-1}{2} \qquad \text{(split the plus/minus sign)}$$

$$= \frac{6}{2} \ \ or \ \ \frac{4}{2} \qquad \text{(simplify the numerators)}$$

$$= \boxed{3 \ \ or \ \ 2} \qquad \text{(and finish up)}$$

There's no homework for this section, but don't be ☹ — there will be plenty of these problems when you study the Quadratic Formula.

Solutions

1. For each quadratic equation, substitute each "solution" (separately) into the original equation. Then work the arithmetic on each side of the equation separately. [Do NOT swap things back and forth across the equals sign.] In all cases, the two sides should balance at the end of the calculations.

2. a. 3, 9, 17 b. 2, –8, 14 c. 6, 2, –2 d. 5, –13, –1
e. 12, 0, 13 f. 1, –13, 0 g. 1, –1, 1 h. –2, 0, –19
i. –1, 14, 0 j. –1, 0, –99 k. 2, –4, –3 l. –3, –2, 5
m. 1, –7, 4 n. 6, –1, –1 o. 18, 0, 0 p. –3, 3, 0

3. a. 145 b. 8 c. 1 d. 25 e. 0 f. –4

4. a. 7 b. –3 c. 7 d. Does not exist e. 8
f. 1 g. 0 h. 6 i. Does not exist j. 9

5. a. –13 b. 5 c. –7 d. –7 e. 1 f. 0

6. a. 121, –121 b. 11, –11 c. 0 d. Does not exist

7. a. 107 b. –16 c. $\frac{7}{2}$ d. –1 e. 0

f. 0 g. $\frac{4}{3}, 0$ h. $\frac{8}{9}, -\frac{10}{9}$

8. a. $b^2 - 4ac$

$= (\ \ )^2 - 4(\ \ )(\ \ )$

Converting each variable to a set of parentheses is a handy way to make sure everything is written properly.

$= (-5)^2 - 4(-10)(3)$

Note: The parentheses around the –5 are required, because as we've learned, $(-5)^2 = 25$, while $-5^2 = -25$.

$= 25 - (-120) = 25 + 120 = 145$

b. 52 c. –140 d. 9

9. a. $\pm\sqrt{b^2 - 4ac}$

$= \pm\sqrt{(\ \ )^2 - 4(\ \ )(\ \ )}$

$= \pm\sqrt{(-7)^2 - 4(2)(-15)}$

$= \pm\sqrt{49 - (-120)}$

$= \pm\sqrt{49 + 120}$

$= \pm\sqrt{169}$

$= \pm 13$

b. 0 c. ± 18 d. ± 3

> "*Education* is
> what you get
> when you read
> the fine print.
> *Experience* is
> what you get if
> you don't."

Preparing for the Quadratic Formula

CH XX –THE QUADRATIC FORMULA, RATIONAL SOLUTIONS

We now have all the tools needed to solve quadratic equations, whether they are factorable or not. In this chapter, our solutions will always be rational numbers; that is, no square roots in the final answers.

☐ *THE QUADRATIC FORMULA*

Here's the formula we create a couple of chapters back:

The quadratic equation $ax^2 + bx + c = 0$

has the solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

☐ EXAMPLES OF USING THE QUADRATIC FORMULA

EXAMPLE 1: **Solve for x: $x^2 - 9x - 10 = 0$**

Solution: We see that this is a quadratic equation in standard form, where $a = 1$, $b = -9$, and $c = -10$. By the Quadratic Formula, x has (possibly) two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-10)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 + 40}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{121}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm 11}{2}$$

Using the plus sign: $x = \frac{9+11}{2} = \frac{20}{2} = 10$

Using the minus sign: $x = \frac{9-11}{2} = \frac{-2}{2} = -1$

Final answer: $\boxed{x = 10, -1}$

Let's check these solutions:

Letting $x = 10$ in the original equation,

$$x^2 - 9x - 10 = \mathbf{10}^2 - 9(\mathbf{10}) - 10 = 100 - 90 - 10 = 0 \checkmark$$

Letting $x = -1$,

$$x^2 - 9x - 10 = (\mathbf{-1})^2 - 9(\mathbf{-1}) - 10 = 1 + 9 - 10 = 0 \checkmark$$

EXAMPLE 2: **Solve for n: $6n^2 + 40 = -31n$**

Solution: The first step is to transform this quadratic equation into standard form. Adding $31n$ to each side (and putting the $31n$ between the $6n^2$ and the 40) produces

$6n^2 + 31n + 40 = 0$ [It's now in standard form.]

Noting that $a = 6$, $b = 31$, and $c = 40$, we're ready to apply the Quadratic Formula:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{(31)^2 - 4(6)(40)}}{2(6)}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{961 - 960}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{1}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm 1}{12}$$

The two solutions are given by

$$\frac{-31+1}{12} = \frac{-30}{12} = -\frac{5}{2}$$

and $\frac{-31-1}{12} = \frac{-32}{12} = -\frac{8}{3}$

$$\boxed{n = -\frac{5}{2}, -\frac{8}{3}}$$

EXAMPLE 3: **Solve for p: $2p^2 = 200$**

Solution: Bringing the 200 over to the left side of the equation gives us our quadratic equation in standard form:

$2p^2 - 200 = 0$

which, if it helps you, can be viewed as

$$2p^2 + \mathbf{0}\boldsymbol{p} - 200 = 0$$

Notice that $a = 2$, $b = 0$, and $c = -200$. Substituting these values into the Quadratic Formula yields our two solutions for p:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad p = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(-200)}}{2(2)}$$

$$\Rightarrow \quad p = \frac{0 \pm \sqrt{0 - 4(2)(-200)}}{4}$$

$$\Rightarrow \quad p = \frac{\pm\sqrt{1600}}{4}$$

$$\Rightarrow \quad p = \frac{\pm 40}{4} = \pm 10$$

Thus, the solutions are $\boxed{p = \pm 10}$

Note: At or near the beginning of this problem, we could have divided each side of the equation by 2. The Quadratic Formula, although containing different values for *a* and *c*, would nevertheless have yielded the same answers, ±10. Try it.

EXAMPLE 4: **Solve for $\boldsymbol{u}$: $\mathbf{5}\boldsymbol{u}^{\mathbf{2}} = \mathbf{9}\boldsymbol{u}$**

Solution: First convert to standard form: $5u^2 - 9u = 0$, from which we deduce that $a = 5$, $b = -9$, and $c = 0$. Plugging these three values into the Quadratic Formula gives:

$$u = \frac{-(\mathbf{-9}) \pm \sqrt{(\mathbf{-9})^2 - 4(\mathbf{5})(\mathbf{0})}}{2(\mathbf{5})} = \frac{9 \pm \sqrt{81 - 0}}{10} = \frac{9 \pm 9}{10}$$

No radicals remain, so that issue is settled; using the plus and minus signs separately yields the two solutions:

$$\frac{9+9}{10} = \frac{18}{10} = \frac{9}{5} \quad \text{and} \quad \frac{9-9}{10} = \frac{0}{10} = 0$$

Final answer: $\boxed{u = \frac{9}{5},\ 0}$

EXAMPLE 5: **Solve for u: $28u - 4u^2 = 49$**

Solution: The standard quadratic form requires that the quadratic term (the squared variable) be in front, followed by the linear term, followed by the constant, all set equal to zero; that is, $ax^2 + bx + c = 0$. We achieve this goal with the following steps:

$$28u - 4u^2 = 49 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad -4u^2 + 28u = 49 \qquad \text{(commutative property)}$$

$$\Rightarrow \quad -4u^2 + 28u - 49 = 0 \qquad \text{(subtract 49 from each side)}$$

At this point, it's in good form for the Quadratic Formula, but it's traditional to not allow the leading coefficient (the –4) to be negative. One way to change –4 into 4 is to multiply it by –1; but, of course, we will have to do this to each side of the equation.

$$\Rightarrow \quad \mathbf{-1}[-4u^2 + 28u - 49] = \mathbf{-1}[0] \quad \text{(multiply each side by –1)}$$

$$\Rightarrow \quad 4u^2 - 28u + 49 = 0 \qquad \text{(distribute)}$$

When written this way,

$$4u^2 - 28u + 49 = 0,$$

and we deduce that $a = 4$, $b = -28$, and $c = 49$. On to the Quadratic Formula:

$$u = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(4)(49)}}{2(4)}$$

$$= \frac{28 \pm \sqrt{784 - 784}}{8}$$

$$= \frac{28 \pm \sqrt{0}}{8} = \frac{28 \pm 0}{8} = \frac{28}{8} = \frac{7}{2}$$

And we're done: $\boxed{u = \frac{7}{2}}$

Notice that although all the previous quadratic equations had two solutions, this equation has just <u>one</u> solution. (Or does it have two solutions that are the same?)

EXAMPLE 6: **Solve for z: $z^2 + 3z + 3 = 0$**

Solution: This quadratic equation seems innocent enough, so let's take the values $a = 1$, $b = 3$, and $c = 3$ and place them in their proper places in the Quadratic Formula:

$$z = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{-3}}{2}$$

Houston . . . we have a problem. Under the assumption that you have not yet encountered imaginary numbers, we cannot consider the square root of a negative number at this point in your studies. If we try to take the square root of –3, the calculator will almost for sure indicate an error. We can't finish this calculation (at least not in the early and middle stages of Algebra), so we must agree that there is simply **NO SOLUTION** to the equation $z^2 + 3z + 3 = 0$.

Homework

1. Solve each quadratic equation:

a. $x^2 - 3x - 10 = 0$

b. $6n^2 + 19n - 7 = 0$

c. $d^2 + 7d = 0$

d. $2u^2 + u + 1 = 0$

e. $z^2 - 49 = 0$

f. $t^2 + 16 = 0$

g. $n^2 - 10n = 0$

h. $w^2 - 144 = 0$

2. Solve each quadratic equation:

a. $n^2 + 10n = -25$

b. $4x^2 = 4x - 1$

c. $3m^2 = 8m$

d. $x^2 = -5 + 3x$

e. $-x^2 - 5x + 6 = 0$

f. $2x^2 - 5x - 3 = 0$

3. Solve each quadratic equation:

a. $2w^2 - 70w + 500 = 0$

b. $w^2 + 2w + 5 = 0$

c. $9w^2 - 24w + 16 = 0$

d. $7n^2 = 10n$

e. $2x^2 - 15x + 7 = 0$

f. $15x^2 - 8 = 2x$

g. $4a^2 + 35a = -24$

h. $21t^2 = 20t + 9$

4. Solve each quadratic equation:

a. $w^2 - 12w + 35 = 0$

b. $2x^2 - 13x + 15 = 0$

c. $n^2 - 20n + 150 = 0$

d. $6a^2 - 31a + 40 = 0$

e. $-2x^2 - 10x + 28 = 0$

f. $25y^2 = 4$

g. $2z^2 + 2 = -4z$

h. $-4a^2 = 4a - 3$

i. $6u^2 = 47u + 8$

j. $0 = 4t^2 + 8t + 3$

k. $-n^2 + n + 56 = 0$

l. $-14x^2 = -40 + 46x$

Review Problems

5. Solve for y: $15y^2 + 2y - 8 = 0$

6. Solve for a: $-49a^2 - 9 = -42a$

7. Solve for x: $4x^2 = 1$

8. Solve for z: $10z^2 = 9z$

9. Solve for n: $3n^2 = 7n - 5$

10. Solve for w: $-2w^2 + 5w + 3 = 0$

☐ *TO ∞ AND BEYOND!*

A. Solve for *x*: $\pi x^2 = \sqrt{2}\,x - \sqrt{\pi}$

B. Solve for *x*: $ax^2 + bx^2 + cx + dx + \pi^2 - \pi = 0$

Solutions

1. a. $5, -2$ b. $\frac{1}{3}, -\frac{7}{2}$ c. $0, -7$ d. No solution
e. ± 7 f. No solution g. $10, 0$ h. ± 12

2. a. -5 b. $\frac{1}{2}$ c. $0, \frac{8}{3}$
d. No solution e. $1, -6$ f. $3, -\frac{1}{2}$

3. a. 25, 10 b. No solution c. $\frac{4}{3}$ d. $0, \frac{10}{7}$

e. $7, \frac{1}{2}$ f. $-\frac{2}{3}, \frac{4}{5}$ g. $-8, -\frac{3}{4}$ h. $\frac{9}{7}, -\frac{1}{3}$

4. a. 5, 7 b. $5, \frac{3}{2}$ c. No solution d. $\frac{8}{3}, \frac{5}{2}$

e. 2, –7 f. $\pm\frac{2}{5}$ g. –1 h. $\frac{1}{2}, -\frac{3}{2}$

i. $8, -\frac{1}{6}$ j. $-\frac{1}{2}, -\frac{3}{2}$ k. 8, –7 l. $-4, \frac{5}{7}$

5. $\frac{2}{3}, -\frac{4}{5}$

6. $\frac{3}{7}$

7. $\pm\frac{1}{2}$

8. $0, \frac{9}{10}$

9. No solution

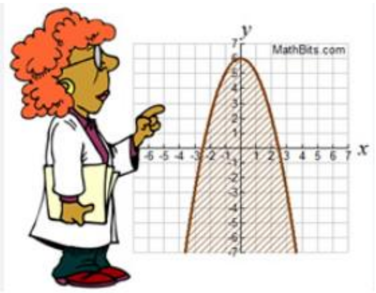
10. $3, -\frac{1}{2}$

“Education's purpose is to replace an empty mind with an open one.”

Malcolm Forbes

CH N – QUADRATIC INEQUALITIES

For linear inequalities such as $7n + 4 \geq 7$, the Basic Principle of Inequalities (If you multiply or divide by a negative number, switch the inequality sign) is sufficient. But other problems are best solved by a technique called the Boundary Point Method. This will help us solve inequalities such as



$$x^2 - 3x - 10 \geq 0.$$

❑ *THE BOUNDARY POINT METHOD*

1) Change the *inequality* to an *equation* and solve the equation.

2) The solutions of that equation are called the *boundary points*.

3) Plot the boundary points on the real line. This will create one or more *intervals* on the line.

4) Now choose a *test point* from each interval (any convenient number in the interval) and plug it into the original inequality. If the test point satisfies the inequality, then the entire interval in which the test point lies is a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is not part of the solution.

5) Lastly, check the boundary points themselves in the inequality.

❑ *SOLVING QUADRATIC INEQUALITIES*

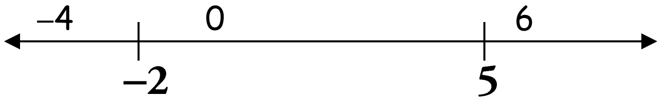
EXAMPLE 1: **Solve for x: $x^2 - 3x - 10 \ge 0$**

Solution: We will solve this kind of problem using the Boundary Point Method (there are other ways). First we find the boundary points:

$$x^2 - 3x - 10 = 0 \Rightarrow (x+2)(x-5) = 0 \Rightarrow x = \mathbf{-2} \text{ or } x = \mathbf{5}.$$

Now we mark these boundary points on a number line, and then choose a test point in each interval to test in the original inequality $x^2 - 3x - 10 \ge 0$:

The **boundary** points are –2 and 5.



The **test** points are –4, 0, and 6.

$\mathbf{-4} \in (-\infty, -2)$ $\mathbf{0} \in (-2, 5)$ $\mathbf{6} \in (5, \infty)$

$(-4)^2 - 3(-4) - 10 = 16 + 12 - 10 = 18$, which is ≥ 0. Thus, $(-\infty, -2)$ is part of the solution.

$0^2 - 3(0) - 10 = 0 - 0 - 10 = -10$, which is not ≥ 0. So, $(-2, 5)$ is not part of the solution.

$6^2 - 3(6) - 10 = 36 - 18 - 10 = 8$, which is ≥ 0. Therefore, $(5, \infty)$ is part of the solution.

Last, you can check the boundary points yourself — they work in the original inequality. Our final answer is the combination of the left interval and the right interval:

$$\boxed{(-\infty, -2] \cup [5, \infty)}$$

The answer can also be written

$$\{x \in \mathbb{R} \mid x \le -2 \text{ OR } x \ge 5\}$$

EXAMPLE 2: **Solve for w: $9w - w^2 \geq 14$**

Solution: Solve the associated equation: $9w - w^2 = 14$. Rearranging the terms and multiplying through by –1 gives the equation $w^2 - 9w + 14 = 0$, whose solutions are 2 and 7.

These two boundary points yield the three intervals to check: $(-\infty, 2)$, $(2, 7)$ and $(7, \infty)$. Choose a test point (your choice) in each interval and substitute that test point into the original inequality. You'll find that the only interval that works is the center one: $(2, 7)$.

Finally, check the boundary points themselves. They both work in the original inequality. The final answer, then, is $\{ w \in \mathbb{R} \mid 2 \leq w \leq 7 \}$, which we write as

$$\boxed{[2, 7]}$$

EXAMPLE 3: **Solve for y: $y^2 - 12 > -4y$**

Solution: To solve this inequality, convert it to an equation, write it in standard quadratic form, factor, and you'll find that the boundary points are –6 and 2. Pick some test points and see that the intervals $(-\infty, -6)$ and $(2, \infty)$ work, but the interval $(-6, 2)$ doesn't. Also, the boundary points themselves don't work. The final answer is therefore

$$\boxed{(-\infty, -6) \cup (2, \infty)}$$

EXAMPLE 4: **Solve for x: $x^2 + x + 4 > 0$**

Solution: The associated equation is $x^2 + x + 4 = 0$, which won't factor, so let's use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(4)}}{2(1)} = \frac{-1 \pm \sqrt{-15}}{2}$$

There are no real solutions to this equation; therefore, there are no boundary points, and so there is only one interval to check, namely the entire real line, $(-\infty, \infty)$.

Choose any real number [that is, any number in the interval $(-\infty, \infty)$], for instance, **2**, as a test point, and plug it into the original inequality: $2^2 + 2 + 4 = 10$, which is clearly > 0. Since the test point worked, the interval in which the test point lies must work; however, this interval was the entire real line. Therefore, the solution to the inequality is the set of real numbers

$$\boxed{\mathbb{R}}$$

EXAMPLE 5: **Solve for x: $x^2 + 6x + 9 < 0$**

Solution: Solving the associated equation, $x^2 + 6x + 9 = 0$, we obtain one solution: $x = -3$. That produces two intervals: $(-\infty, -3)$ and $(-3, \infty)$. Pick a test point in each interval and you'll discover that both test points fail; even the boundary point $x = -3$ fails. In other words, everything fails! The inequality has **NO solution** at all. We can write this using the null set (empty set):

$$\boxed{\varnothing}$$

EXAMPLE 6: **Solve for x: $x^2 + 6x + 9 \le 0$**

Solution: Solving the associated equation gives us the same boundary point as in the previous example. The two intervals also fail, as in the previous example. BUT this time, if we check the boundary point, $x = -3$, it works. So our final solution of the inequality is

$\boxed{x = -3}$ [which can also be written $\{-3\}$.]

Homework

Solve each quadratic inequality:

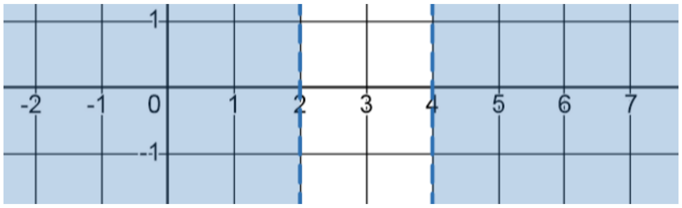
1. $a^2 - 9a - 10 > 0$
2. $2x^2 \le 5 - 9x$
3. $x^2 + 9 < 6x$
4. $y^2 + 10y + 25 \le 0$
5. $z^2 > 12z - 36$
6. $w^2 + 14w + 49 \ge 0$
7. $x^2 \ge 9$
8. $2x^2 - 15x + 7 < 0$
9. $5a - a^2 \ge 6$
10. $-3(3n^2 + 8n) \le 16$

❑ *USING DESMOS TO SOLVE QUADRATIC INEQUALITIES*

You can use Desmos to solve these problems. For example, type

$$x^2 - 6x + 8 > 0$$

and you'll get the following graph:



Look at the two shaded regions. We see shading to the left of 2 and to the right of 4. This means that the inequality is satisfied whenever $x < 2$ or $x > 4$, giving us a final answer of

$$(-\infty, -2) \cup (4, \infty)$$

Review Problems

Solve each inequality:

11. $2(x - 5) - 9(2x + 1) > 0$

12. a. $x^2 + 7x + 12 < 0$ b. $x^2 - 100 \ge 0$
 c. $x^2 + 10x + 25 > 0$ d. $x^2 - 6x + 9 \le 0$
 e. $4x^2 - 12x + 9 < 0$ f. $x^2 \ge 2x - 1$

13. $x^2 - 5x \ge -4$

14. $x^2 < 20x - 100$

Solutions

1. $(-\infty, -1) \cup (10, \infty)$

2. $\left[-5, \frac{1}{2}\right]$

3. $\varnothing$

4. $\{-5\}$

5. $(-\infty, 6) \cup (6, \infty)$; or $\mathbb{R} - \{6\}$

6. $\mathbb{R}$

7. $(-\infty, -3] \cup [3, \infty)$

8. $\left(\frac{1}{2}, 7\right)$

9. $[2, 3]$

10. $\mathbb{R}$

11. $\left(-\infty, -\frac{19}{16}\right)$

12. a. $(-4, -3)$ b. $(-\infty, -10] \cup [10, \infty)$ c. $\mathbb{R} - \{-5\}$
d. $\{3\}$ e. $\varnothing$ f. $\mathbb{R}$

13. $(-\infty, 1] \cup [4, \infty)$

14. $\varnothing$

“A child
without education
is like a bird
without wings.”

—Tibetan Proverb

CH XX – QUADRATIC MODELING, PART I

Whether you're trying to maximize your profits in a factory, or trying to design a cattle corral of maximum area with a given amount of fence, much of economics and engineering involves making the most of limited resources. The vertex of a parabola will allow us to solve such optimization problems.

☐ *SHORTCUT TO THE VERTEX*

Finding the vertex of a parabola by averaging the x-coordinates of the x-intercepts of the parabola is the same time-consuming process over and over again. Perhaps we can find the vertex of a generic parabola, yielding a simple formula that can find the vertex of any parabola quickly.

Let's start with a generic parabola (opening up or down) in standard form:

$$y = ax^2 + bx + c \qquad \text{[opens } up \text{ if } a > 0 \text{ and } down \text{ if } a < 0]$$

Our job now is to find the x-intercepts of this parabola. To find any x-intercepts, we set y to 0 and solve for x:

$$0 = ax^2 + bx + c$$

Let's first turn the equation around so it's in standard form:

$$ax^2 + bx + c = 0$$

How do we solve this equation for x? Actually, it's quite simple. The solutions of this equation are precisely the solutions provided by the Quadratic Formula. Thus, the two solutions of this equation are

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

These two "numbers" are the x-coordinates of the x-intercepts of the parabola. The next step is to find the average of these two numbers; this is done by adding them together and dividing by 2:

$$\frac{\dfrac{-b+\sqrt{b^2-4ac}}{2a} + \dfrac{-b-\sqrt{b^2-4ac}}{2a}}{2}$$

$$= \frac{\dfrac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}}{2} \qquad \text{(common denominator)}$$

$$= \frac{\dfrac{-2b}{2a}}{2} \qquad \text{(the radicals sum to 0)}$$

$$= \frac{\dfrac{-b}{a}}{2} \qquad \text{(reduce the top fraction)}$$

$$= \frac{-b}{a}\cdot\frac{1}{2} \qquad \text{(multiply by reciprocal)}$$

$$= \frac{-b}{2a}$$

We've proved a theorem:

The x-coordinate of the **vertex** of the parabola

$$y = ax^2 + bx + c$$

is given by the formula

$$x = \frac{-b}{2a}$$

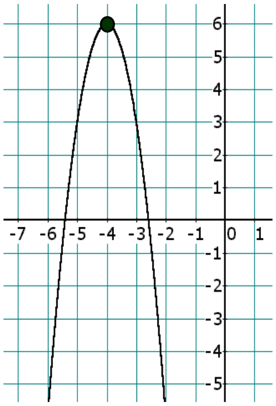
To find the *y*-coordinate of the vertex, just plug the *x*-coordinate of the vertex into the parabola formula.

For example, let's find the vertex of the parabola $y = -3x^2 - 24x - 42$. For this parabola, $a = -3$ and $b = -24$ (we don't need the value of c for this problem). So the *x*-coordinate of the parabola is

$$x = \frac{-b}{2a} = \frac{-(-24)}{2(-3)} = \frac{24}{-6} = -4$$

Now plug $x = -4$ into the parabola formula to find the *y*-coordinate of the vertex:

$$y = -3x^2 - 24x - 42 = -3(\mathbf{-4})^2 - 24(\mathbf{-4}) - 42$$
$$= -3(16) - 24(-4) - 42 = -48 + 96 - 42 = \mathbf{6}$$



Conclusion: The vertex of the parabola $y = -3x^2 - 24x - 42$ is **(–4, 6)**.

> Here's a good way to remember this vertex formula: Take a look at the Quadratic Formula, cross out the radical, and what remains is the *x*-coordinate of the vertex.

$$x = \frac{-b \cancel{\pm \sqrt{b^2 - 4ac}}}{2a}$$

Homework

1. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = x^2 + 2x - 48$

b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$

d. $y = 3x^2 - 6x + 4$

e. $y = -5x^2 + 40x - 20$

f. $y = -3x^2 - 16$

2. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = -2x^2 + 5x - 7$

b. $y = 7x^2 - x - 1$

c. $y = 3x^2 - 3x - 2$

d. $y = 7x^2 + 10$

e. $y = 4x^2 + 7x$

f. $y = -5x^2 - 14x + 10$

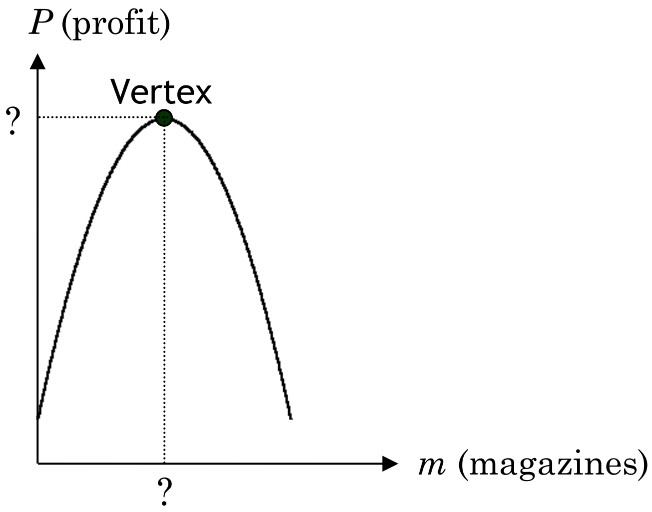
☐ *OPTIMIZATION PROBLEMS*

EXAMPLE 1: **A publishing company found that the profit they make on their latest magazine is given by the formula**

$$\boldsymbol{P = -2m^2 + 680m + 10200}$$

where P is profit and m is the number of magazines sold. Find the number of magazines that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

Solution: Wow, what a problem! It might help to realize that the variable m is playing the role of the variable x, while the P is playing the role of y. When viewed this way, the profit formula is simply a parabola that opens down (since $-2 < 0$). And where is the maximum point on a parabola that opens down? At its vertex, of course.



Once we find the vertex of the given parabola formula, we can find the number of magazines needed to maximize the profit, and then the dollar amount of the maximum profit itself.

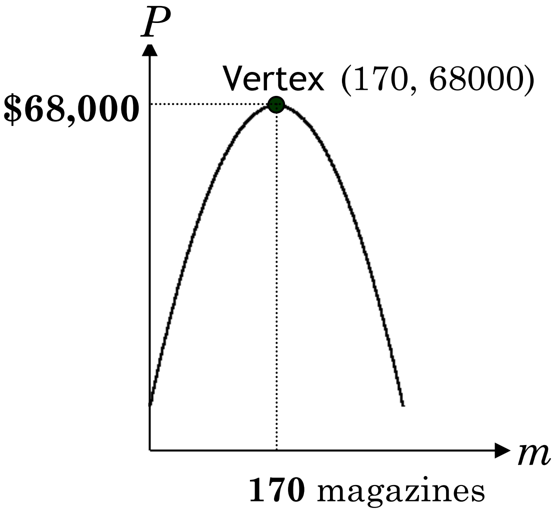
From the given parabola formula $P = -2m^2 + 680m + 10200$, we see that $a = -2$ and $b = 680$. Plugging these two values into our vertex formula gives us the m-coordinate of the vertex:

$$m = \frac{-b}{2a} = \frac{-680}{2(-2)} = \frac{-680}{-4} = \mathbf{170}$$

And since the m-axis is the horizontal axis (the magazine axis), we conclude that the number 170 represents magazines. That is, 170 magazines will produce the maximum profit. To calculate that profit, we merely let $m = 170$ in the profit formula:

$$\begin{aligned} P &= -2m^2 + 680m + 10200 && \text{(the profit formula)} \\ &= -2(\mathbf{170})^2 + 680(\mathbf{170}) + 10200 && (m = 170) \\ &= -2(28900) + 680(170) + 10200 && \text{(exponents first)} \\ &= -57800 + 115600 + 10200 && \text{(multiplications second)} \\ &= \mathbf{68000} && \text{(additions last)} \end{aligned}$$

The vertex of the parabola is therefore the point (170, 68000).



In conclusion, producing **170** magazines will yield the company a maximum profit of **$68,000**.

EXAMPLE 2: **A company found that the cost for manufacturing a surfboard is given by the formula**

$$\boldsymbol{C = 3s^2 - 270s + 6800}$$

where C is the cost in dollars and s is the number of surfboards produced. Find the number of boards the company needs to produce in order to minimize their cost. What is the minimum cost?

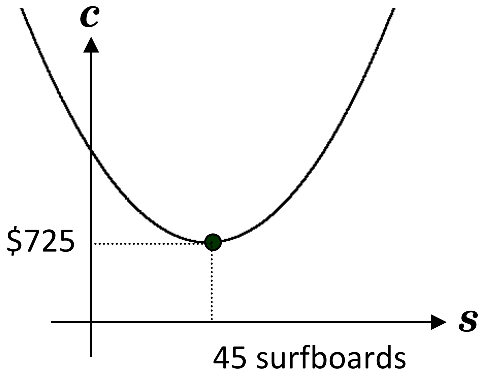
Solution: This problem is very similar to Example 1, except here we are to find the minimum of something, not the maximum. The graph of the cost formula is a parabola opening up (since 3 > 0), so its vertex will be at the bottom of the parabola, and, as before, the vertex tells us everything we need to know.

Since $a = 3$ and $b = -270$, the s-coordinate of the vertex is

$$s = \frac{-b}{2a} = \frac{-(-270)}{2(3)} = \frac{270}{6} = \mathbf{45}$$

Plugging this value of s into the cost formula yields c = **725**. Therefore, the vertex of the parabola is **(45, 725)**.

What does all this mean? Geometrically, the point (45, 725) is at the bottom of the parabola. Therefore,



45 surfboards will result in a minimum cost of **$725**.

Homework

3. A hard drive company found that the profit they make on their hard drive is given by the formula

$$P = -4h^2 + 200h + 1000,$$

where P is profit and h is the number of hard drives sold. Find the number of hard drives that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

4. A company found that the cost for manufacturing a flash drive is given by the formula

$$C = 7d^2 - 140d + 745,$$

where C is the cost in dollars and d is the number of flash drives produced. Find the number of flash drives the company needs to produce in order to minimize their cost. What is the minimum cost?

5. The path of a baseball follows the parabola

$$y = -12x^2 + 456x + 100$$

At what value of x does the baseball reach its highest point? How high above the x-axis is the highest point?

6. The profit, P, is given by the formula

$$P = -10c^2 + 800c + 10000$$

Find the number of calculators, c, which would maximize the profit, and determine the maximum profit.

7. The cost, C, for manufacturing w widgets is given by

$$C = 7w^2 - 112w + 5000$$

How many widgets will minimize the cost, and what is the cost?

8. The path of a catapulted math teacher follows the parabola $y = -12x^2 + 312x + 250$. Find the point in the x-y plane where the teacher reaches his highest point.

9. A certain mountain has the approximate shape of the parabola $y = -13x^2 + 390x + 200$. Find the coordinates of the mountain peak.

10. A certain valley has the approximate shape of the parabola $y = 7.5x^2 - 1485x - 400$. Find the coordinates of the lowest point of the valley.

Review Problems

11. Using the short formula from this chapter, find the vertex of the parabola

    $$y = -5x^2 + 70x - 90.$$

12. The profit is given by the formula

    $$P = -5b^2 + 750b + 10000$$

    where P is profit and b is the number of books published. Find the number of books which would maximize the profit, and determine the maximum profit.

13. The cost, C, for manufacturing t trumpets is given by

    $$C = 6t^2 - 276t + 5000$$

    How many trumpets will minimize the cost, and what is the cost?

14. The path of a football follows the parabola $y = -7x^2 + 126x + 250$. Find the point in the x-y plane where the football reaches its highest point.

Solutions

1. a. $(-1, -49)$ b. $(-5, 0)$ c. $(-2, -3)$
d. $(1, 1)$ e. $(4, 60)$ f. $(0, -16)$

2. a. $\left(\frac{5}{4}, -\frac{31}{8}\right)$ b. $\left(\frac{1}{14}, -\frac{29}{8}\right)$ c. $\left(\frac{1}{2}, -\frac{11}{4}\right)$

d. (0, 10) e. $\left(-\frac{7}{8}, -\frac{49}{16}\right)$ f. $\left(-\frac{7}{5}, \frac{99}{5}\right)$

3. 25 hard drives; maximum profit = \$3,500

4. 10 flash drives; minimum cost = \$45

5. x = 19; height = 4,432

6. 40 calculators; maximum profit = \$26,000

7. 8 widgets; minimum cost = \$4552

8. (13, 2278) **9**. (15, 3125)

10. (99, –73907.5) **11**. V(7, 155)

12. 75 books; \$38,125 **13**. 23 trumpets; \$1826

14. (9, 817)

“The purpose of learning is growth, and our minds, unlike our bodies, can continue growing as we continue to live.”

– Mortimer Adler

CH NN – QUADRATIC MODELING, PART II

As indicated by the chapter title, we continue our discussion of the uses of quadratic functions (parabolas) to solve business problems involving both revenue and expenses. We'll also solve problems involving the maximization of the area of a rectangular region using a given amount of fencing.

EXAMPLE 1: **The revenue function at Widget, Inc., where w is the number of widgets produced, is given by $R(w) = 2w^2 + 100w - 6$, while the expense function is $E(w) = 7w^2 - 50w + 1$. Find the production level of widgets that will yield a maximum profit, and then calculate the profit.**

Solution: Revenue is the money received from selling a product or service, and expense is the cost incurred in providing that product or service to the customer. Profit is the difference between revenue and expenses, so we can write the formula

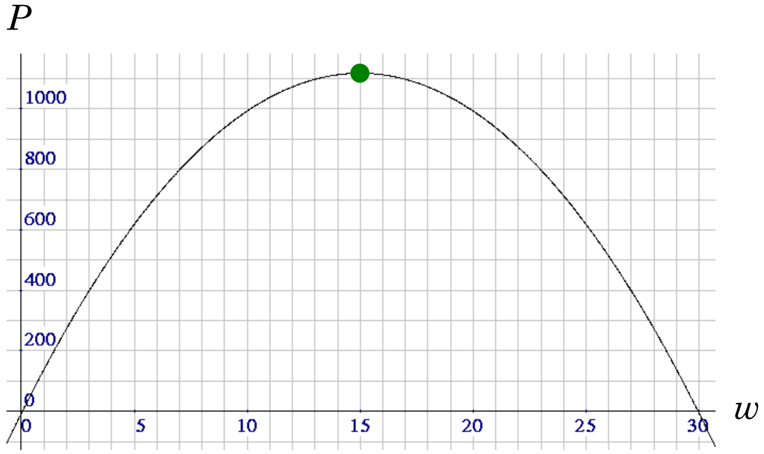
$$P(w) = R(w) - E(w)$$

$$\Rightarrow \quad P(w) = (2w^2 + 100w - 6) - (7w^2 - 50w + 1)$$

$$\Rightarrow \quad P(w) = 2w^2 + 100w - 6 - 7w^2 + 50w - 1$$

$$\Rightarrow \quad P(w) = -5w^2 + 150w - 7$$

This formula shows that the profit, P, is a quadratic function of the number of widgets, w. So its graph is a parabola opening down (since $-5 < 0$). Because we're trying to determine the maximum profit, we observe that the maximum value of P will therefore be at the vertex (top) of the parabola. A rough sketch of the profit function (the parabola) should make this concept clear.



The w-coordinate of the vertex is given by the formula $\frac{-b}{2a}$, where $a = -5$ and $b = 150$. So, $w = \frac{-b}{2a} = \frac{-150}{2(-5)} = \frac{-150}{-10} = 15$.

Hence, the production level of widgets should be 15. This maximizes the profit. Fewer than 15 widgets would result in less profit; more than 15 widgets would also reduce profit. The optimum number of widgets is 15.

Now to calculate the profit on the 15 widgets:

$$P(w) = -5w^2 + 150w - 7$$

$$\Rightarrow \quad P(\mathbf{15}) = -5(\mathbf{15})^2 + 150(\mathbf{15}) - 7$$

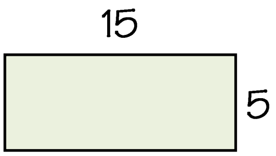
$$\Rightarrow \quad P(15) = 1118$$

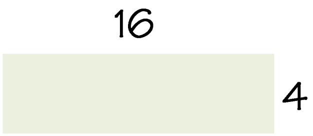
Producing **15** widgets gives a maximum profit of **$1,118**

This result is consistent with the graph of the parabola.

EXAMPLE 2: **Farmer Gertrude has 40 feet of fence to build a rectangular pigpen. What should the dimensions of the pigpen be so that it will enclose the maximum area? What will the area be with these dimensions?**

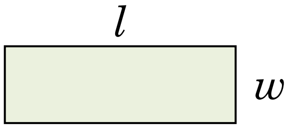
Solution: As a student I remember thinking that the given 40 feet of fence would produce the same area regardless of the dimensions of the rectangle. But consider:





Each rectangle has the same perimeter (the 40 feet of fence), but the areas (75 sq ft and 64 sq ft) are different — the first rectangle contains 11 additional sq ft, perhaps room for another pig.

The solution begins by labeling a rectangular pigpen:



Now, the perimeter of the rectangle is given to be 40 feet. Thus,

$$2l + 2w = 40$$ (this equation is called the **constraint**)

Also, the area of the rectangle is to be made as large as possible.

$$A = lw$$ (this is called the **objective function**)

We'd like to graph this area function, but the problem with this formula is that the area is a function of too many variables (we can't graph three variables in a two-dimensional coordinate system). But we can tie the perimeter formula to the area formula with a little substitution:

Solve for l in the perimeter formula:

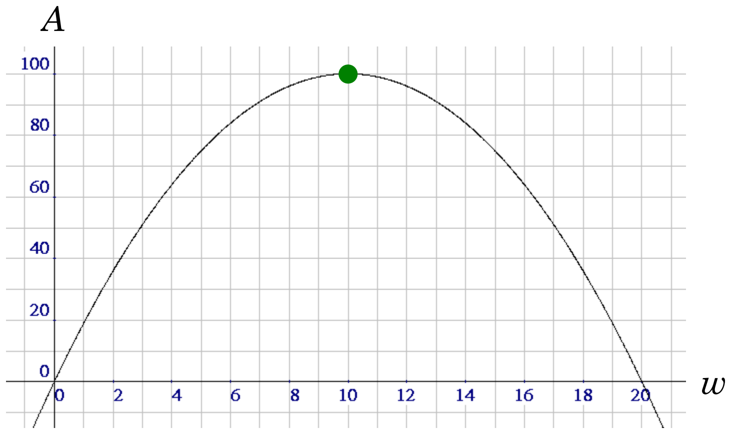
$$2l + 2w = 40 \quad \Rightarrow \quad 2l = 40 - 2w$$

$$\Rightarrow \quad l = \frac{40 - 2w}{2} \quad \Rightarrow \quad \underline{l = 20 - w}$$

Now substitute $20 - w$ for l in the area formula:

$$\begin{array}{llll} & A = lw & & \text{(the area formula)} \\ \Rightarrow & A = (\underline{20 - w})w & & \text{(letting } l = 20 - w\text{)} \\ \Rightarrow & A = 20w - w^2 & & \text{(distribute)} \\ \Rightarrow & A = -w^2 + 20w & & \text{(better form)} \end{array}$$

We have written the area as a function of just one variable. We want to find the value of w that will maximize A. We can find this value of w because the equation we just wrote is a parabola.



Looking at the graph, we conjecture that the vertex — the maximum point on the curve — occurs at the point (10, 100). This is verified by using the formula for the vertex, where the w-coordinate is given by

$$w = \frac{-b}{2a} = \frac{-20}{2(-1)} = \frac{-20}{-2} = 10$$

Thus, the width is 10 feet. Using the formula $l = 20 - w$ we see that the length is also 10 feet. The rectangle of maximum area is therefore actually a

$$\boxed{10 \text{ ft} \times 10 \text{ ft square}}$$

whose area is 100 sq ft.

Homework

1. The revenue function is $R(x) = 2x^2 - 5x - 25$, while the expense function is $E(x) = 7x^2 - 85x + 75$. Find the production level that will yield a maximum profit. What is the profit?
   Recall: Profit = Revenue – Expenses.

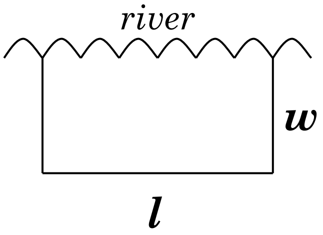
2. The revenue function is $R(x) = 2x^2 + 2000x + 200$, while the expense function is $E(x) = 11x^2 + 596x - 50$. Find the production level that will yield a maximum profit. What is the profit?

3. To protect his corn from Farmer Gertrude's pigs (Gertrude made the 10 ft × 10 ft pigpen alright, but she made it only 9 inches tall!), Uncle Eb needs to enclose his corn in a rectangular region, using just 84 feet of fence. He would like as much area as he can get with his fence. What should be the dimensions of Uncle Eb's cornfield?

4. Same as the previous problem, but assume 180 feet of fence.

5. Rancher Millie liked what Gertrude and Eb had done, so when she needed to build a cow corral, she consulted them about her problem. Gertrude and Eb told Millie that a square would definitely be the best shape, since a square gave both of them the most area for the given amount of fence. What Millie didn't think was relevant to tell her friends was that a river would be used as one

side of the corral; that is, only three sides of the rectangle would have to be fenced:



If Rancher Millie has 1000 feet of fence to build the 3-sided rectangle, what dimensions would yield her the maximum area?

6. The same scenario as the previous problem, except Rancher Millie has 1200 feet of fence.

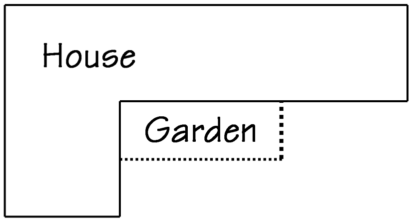
Review Problems

7. The revenue function for the widget factory is $R(w) = 23w^2 + 950w + 95$, and the expense function is $E(w) = 33w^2 + 350w - 5$. How many widgets should be produced to achieve the maximum profit?

8. What dimensions should a rectangular region be if the perimeter must be 100 and if the maximum area is desired?

9. A rectangular corral is to be built with 144 feet of fence, using the side of a barn as one of the sides of the rectangle. If the goal is to enclose the maximum number of square feet, what should the dimensions of the rectangle be?

10. Bruce has 60 feet of fence with which to enclose a rectangular region for his flower garden. But an inside corner of the yard will provide two of the four sides of the rectangle. Find the dimensions of the rectangle that will provide Bruce with the maximum possible area for his garden.



Solutions

1. 8 items; $220

2. 78 items; $55,006

3. 21 ft × 21 ft

4. 45 ft × 45 ft

5. Hint: It's not a square

6. 600 ft × 300 ft

7. 30 widgets

8. 25 × 25

9. 72 ft × 36 ft

10. 30 ft × 30 ft

8

Education is not

a preparation for life;

education is life itself.

John Dewey

CH X – BEYOND SQUARE ROOTS

☐ *SQUARE ROOTS REVIEW*

The number 9 has two ***square roots***, 3 and –3. This is because $3^2 = 9$ and $(-3)^2 = 9$. The positive square root of 9 (the 3) is denoted $\sqrt{9}$, and the negative square root of 9 (the –3) is written $-\sqrt{9}$. In other words, $\sqrt{9} = 3$, and only 3, while $-\sqrt{9} = -3$.

In analyzing $\sqrt{-25}$, the "principle" square root of –25, we discover that we cannot find an answer for this problem, since the square of a real number can never be negative. If there is an answer to $\sqrt{-25}$, it lies outside ℝ, the set of real numbers.

Sometimes the square root radical is written $\sqrt[2]{x}$, with a little 2 cradled in the radical sign. The 2 refers to the fact that taking the square root is essentially the opposite of squaring.

☐ *HIGHER ROOTS*

Consider the number 8. Since two cubed is eight, $2^3 = 8$, we can say that 2 is a ***cube root*** of 8. In fact, it's the only cube root of 8, simply because there's no other real number whose cube is 8.
We write $\sqrt[3]{8} = 2$. Perhaps a little surprising is that we can calculate the cube root of a negative number without leaving the real numbers, ℝ. For example, $\sqrt[3]{-27}$ equals –3, since

$$(-3)^3 = -27 \qquad [-3 \cdot -3 \cdot -3 = -27]$$

The number 16 has <u>two</u> ***fourth roots***. The positive fourth root is $\sqrt[4]{16} = 2$, and the negative fourth root is $-\sqrt[4]{16} = -2$. After all, both 2 and –2 raised to the fourth power result in 16. However, just like square roots, $\sqrt[4]{-1}$ is not a real number.

The ***fifth root*** of 32 is 2; that is, $\sqrt[5]{32} = 2$. This is because $2^5 = 32$. Like cube roots, we can calculate the fifth root of a negative number. For example, $\sqrt[5]{-243}$ equals –3, since $(-3)^5 = -243$.

Homework

1. Find the square root(s) of

   a. 100 b. 15 c. 0 d. –36 e. 1

2. Find the cube root(s) of

   a. 64 b. –125 c. 0 d. 20 e. 1

3. Find the fourth root(s) of

   a. 81 b. 0 c. –625 d. 25 e. 1

4. Find the fifth root(s) of

   a. 1 b. 0 c. –243 d. 29 e. 32

5. Evaluate each radical:

   a. $\sqrt{169}$ b. $\sqrt{225}$ c. $\sqrt[3]{8}$ d. $\sqrt[3]{27}$ e. $\sqrt[3]{-125}$

   f. $\sqrt[4]{625}$ g. $\sqrt[4]{1}$ h. $\sqrt[4]{-16}$ i. $\sqrt[5]{-32}$ j. $\sqrt[5]{0}$

   k. $\sqrt[3]{64}$ l. $\sqrt[3]{216}$ m. $\sqrt[3]{-64}$ n. $-\sqrt[5]{-1}$ o. $\sqrt[4]{16}$

   p. $-\sqrt[4]{81}$ q. $\sqrt[3]{-1}$ r. $-\sqrt[4]{-1}$ s. $\sqrt[5]{243}$ t. $\sqrt{0} + \sqrt[3]{0}$

☐ SIMPLIFYING MORE ROOTS

Assume that x and y represent non-negative numbers. Then

$$\boxed{\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}}$$

The n*th* root of a product
is the product of the n*th* roots

Another useful rule for radicals is the following. If $\boldsymbol{x}$ represents a non-negative number ($\boldsymbol{x} \geq 0$), then

$$\boxed{\sqrt[n]{x^n} = x}$$

The *nth* root cancels
the *nth* power

[The special case of simplifying $\sqrt{x^2}$, where x has negative value, will not be dealt with in this chapter.]

EXAMPLES: **Simplify each radical expression:**

A. $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

B. $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27} \times \sqrt[3]{3} = 3\sqrt[3]{3}$

C. $\sqrt[4]{1250} = \sqrt[4]{625 \cdot 2} = \sqrt[4]{625} \cdot \sqrt[4]{2} = 5\sqrt[4]{2}$

D. $\sqrt[3]{2250}$ Sometimes the radicand (the 2250) is too big to easily see if there's a perfect cube in it. So let's try a slightly different approach. We factor the 2250 into primes to get

$$2250 = 2 \cdot 3^2 \cdot 5^3$$

Clearly we can take the cube root of 5^3 (it's 5), but there are not enough of the other factors to take their cube roots. So we can write

$$\sqrt[3]{2250} = \sqrt[3]{2 \cdot 3^2 \cdot 5^3} = \sqrt[3]{5^3} \cdot \sqrt[3]{2 \cdot 3^2} = 5\sqrt[3]{18}$$

Homework

6. Simplify each radical:

a. $\sqrt{288}$
b. $\sqrt[3]{54}$
c. $\sqrt[3]{16}$
d. $\sqrt[3]{250}$
e. $\sqrt[4]{32}$
f. $\sqrt[4]{243}$
g. $\sqrt[4]{162}$
h. $\sqrt[4]{1}$
i. $\sqrt[3]{-54}$
j. $\sqrt[4]{-16}$
k. $\sqrt[5]{64}$
l. $\sqrt[5]{486}$
m. $\sqrt[3]{135}$
n. $\sqrt[4]{162}$
o. $\sqrt[3]{189}$
p. $\sqrt[5]{96}$
q. $\sqrt[3]{128}$
r. $\sqrt[4]{1250}$
s. $\sqrt[3]{250}$
t. $\sqrt[3]{432}$
u. $\sqrt[5]{320}$
v. $\sqrt[3]{48}$
w. $\sqrt[4]{648}$
x. $\sqrt[5]{2673}$

Review Problems

7. Simplify each radical:

a. $\sqrt[3]{108}$ b. $\sqrt[4]{405}$ c. $\sqrt[5]{192}$ d. $\sqrt[3]{-500}$

e. $\sqrt[4]{-32}$ f. $\sqrt[5]{-486}$ g. $\sqrt[3]{3000}$ h. $\sqrt[4]{567}$

i. $\sqrt[3]{648}$ j. $\sqrt[5]{320}$ k. $\sqrt[3]{81}$ l. $\sqrt[3]{250}$

m. $\sqrt[3]{-56}$ n. $\sqrt[4]{48}$ o. $\sqrt[4]{-405}$ p. $\sqrt[4]{768}$

q. $\sqrt[5]{128}$ r. $\sqrt[5]{1215}$

Solutions

1.	a. ± 10	b. $\pm\sqrt{15}$	c. 0	d. Not real	e. ± 1
2.	a. 4	b. -5	c. 0	d. $\sqrt[3]{20}$	e. 1
3.	a. ± 3	b. 0	c. Not real	d. $\pm\sqrt[4]{25}$	e. ± 1
4.	a. 1	b. 0	c. -3	d. $\sqrt[5]{29}$	e. 2
5.	a. 13	b. 15	c. 2	d. 3	e. -5
	f. 5	g. 1	h. Not real	i. -2	j. 0
	k. 4	l. 6	m. -4	n. 1	o. 2
	p. -3	q. -1	r. Not real	s. 3	t. 0

6. a. $12\sqrt{2}$ b. $3\sqrt[3]{2}$ c. $2\sqrt[3]{2}$ d. $5\sqrt[3]{2}$ e. $2\sqrt[4]{2}$ f. $3\sqrt[4]{3}$
g. $3\sqrt[4]{2}$ h. 1 i. $-3\sqrt[3]{2}$ j. Not real k. $2\sqrt[5]{2}$ l. $3\sqrt[5]{2}$
m. $3\sqrt[3]{5}$ n. $3\sqrt[4]{2}$ o. $3\sqrt[3]{7}$ p. $2\sqrt[5]{3}$ q. $4\sqrt[3]{2}$ r. $5\sqrt[4]{2}$
s. $5\sqrt[3]{2}$ t. $6\sqrt[3]{2}$ u. $2\sqrt[5]{10}$ v. $2\sqrt[3]{6}$ w. $3\sqrt[4]{8}$ x. $3\sqrt[5]{11}$

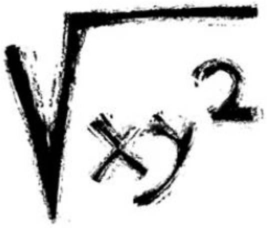
7. a. $3\sqrt[3]{4}$ b. $3\sqrt[4]{5}$ c. $2\sqrt[5]{6}$ d. $-5\sqrt[3]{4}$ e. Not real
f. $-3\sqrt[5]{2}$ g. $10\sqrt[3]{3}$ h. $3\sqrt[4]{7}$ i. $6\sqrt[3]{3}$ j. $2\sqrt[5]{10}$
k. $3\sqrt[3]{3}$ l. $5\sqrt[3]{2}$ m. $-2\sqrt[3]{7}$ n. $2\sqrt[4]{3}$ o. Not real
p. $4\sqrt[4]{3}$ q. $2\sqrt[5]{4}$ r. $3\sqrt[5]{5}$

> *"An educational system isn't worth a great deal if it teaches young people how to make a living– but doesn't teach them how to make a life."*

– Unknown

CH XX – MORE OPERATIONS ON RADICALS

The chapter on fractional exponents showed us that *x* to the power 1/*n* is equal to the *n*th root of *x*:



$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

This way of expressing radicals using exponents allows us to prove a pair of rules we previously took for granted.

☐ *PROVING THE THEOREMS ABOUT RADICALS*

THEOREM: Assume that x and y represent non-negative numbers. Then

$$\boxed{\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}}$$

The root of a product
is the product of the roots.

Proof: Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

$$\sqrt[n]{xy}$$

$$= \quad (xy)^{1/n} \qquad \text{(definition of fractional exponent)}$$

$$= \quad x^{1/n}\,y^{1/n} \qquad \text{(law of exponents)}$$

$$= \quad \sqrt[n]{x}\,\sqrt[n]{y} \qquad \text{(definition of fractional exponent)}$$

THEOREM: If x represents a non-negative number (that is, if $\boldsymbol{x \ge 0}$), then

$$\boxed{\sqrt[n]{x^n} \;=\; x}$$

The *n*th root cancels
the *n*th power.

Proof: Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} \;=\; (x^n)^{1/n} \;=\; x^{n\cdot\frac{1}{n}} \;=\; x^1 \;=\; x$$

☐ *ADDING AND SUBTRACTING RADICALS*

We have a simple radical multiplication rule which states that $\sqrt{a}\sqrt{b} \;=\; \sqrt{ab}$. How about a rule like $\sqrt{a} \;+\sqrt{b} \;=\; \sqrt{a+b}$? Seems reasonable; let's perform an experiment. Let $a = 16$ and $b = 9$.

Left side: $\sqrt{a}+\sqrt{b} \;=\; \sqrt{16}+\sqrt{9} \;=\; 4+3 \;=\; \mathbf{7}$

Right side: $\sqrt{a+b} \;=\; \sqrt{16+9} \;=\; \sqrt{25} \;=\; \mathbf{5}$ ☹

We got two different results; we thus conclude that $\sqrt{a}+\sqrt{b} \;\neq\; \sqrt{a+b}$. [Of course, if $a = 0$ or $b = 0$, then it does work; but if it doesn't always work, it's not a rule. It's considered a false statement.]

So maybe we can't add $\sqrt{7}$ and $\sqrt{11}$ together, but we <u>can</u> add $2\sqrt{7}$ and $3\sqrt{7}$ together. This is simply the *combining of like terms*. After all, we may not be very comfortable with a number like $\sqrt{7}$, but certainly 2 of them plus 3 more of them should total 5 of them: $2\sqrt{7}+3\sqrt{7} = 5\sqrt{7}$.

Alternatively, we can apply the distributive property to the sum $2\sqrt{7}+3\sqrt{7}$ by factoring out the GCF: $\sqrt{7}(2+3)$, again giving $5\sqrt{7}$.

Thus, to add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

<u>EXAMPLE 1:</u> **Simplify each radical expression:**

A. $3\sqrt{11}+5\sqrt{11} = 8\sqrt{11}$

They're like terms, so we can add them up, just like $3x + 5x = 8x$.
Or, factor out the GCF: $\sqrt{11}\left(3+5\right) = 8\sqrt{11}$

B. $18\sqrt[3]{10}-20\sqrt[3]{10} = -2\sqrt[3]{10}$

C. $\sqrt{3}+\sqrt{5}$ cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D. $3\sqrt{20}+10\sqrt{45}$ It might appear that these terms cannot be combined, but don't jump to conclusions — a little simplification first will actually yield an answer:

$$
\begin{aligned}
& 3\sqrt{20}+10\sqrt{45} \\
= \quad & 3\sqrt{4\cdot 5}+10\sqrt{9\cdot 5} \\
= \quad & 3\cdot 2\sqrt{5}+10\cdot 3\sqrt{5} \\
= \quad & 6\sqrt{5}+30\sqrt{5} && \text{[Like Terms!]} \\
= \quad & 36\sqrt{5}
\end{aligned}
$$

E. $\sqrt{7}+\sqrt[3]{7}$ can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

Homework

1. Simplify each radical expression:

a. $2\sqrt{x}+10\sqrt{x}$

b. $3\sqrt{7}-2\sqrt{2}$

c. $10\sqrt[3]{n}+3\sqrt[3]{n}$

d. $6\sqrt{x}-2\sqrt[3]{x}$

e. $9\sqrt{8}+10\sqrt{50}$

f. $\sqrt[3]{24}+\sqrt[3]{81}$

g. $3\sqrt{80}-5\sqrt{45}$

h. $10\sqrt[3]{189}+\sqrt[3]{56}$

i. $\sqrt[3]{z^2}+2\sqrt[3]{z^2}$

j. $\sqrt{12}+2\sqrt{75}$

k. $2\sqrt{20}+2\sqrt{45}$

l. $2\sqrt{275}-3\sqrt{99}$

☐ *MULTIPLYING RADICALS*

Using the formula $\sqrt[n]{x}\cdot\sqrt[n]{y} = \sqrt[n]{xy}$, we can multiply "like roots."

<u>EXAMPLE 2:</u> **Simplify each radical expression:**

A. $\sqrt[5]{3}\cdot\sqrt[5]{7} = \boldsymbol{\sqrt[5]{21}}$

B. $\sqrt{7}\sqrt{7} = \sqrt{49} = \mathbf{7}$

C. $\sqrt[3]{17}\times\sqrt[3]{3} = \boldsymbol{\sqrt[3]{51}}$

D. $\sqrt{2}\sqrt{8} = \sqrt{16} = \mathbf{4}$

Alternate Method: $\sqrt{2}\sqrt{8} = \sqrt{2}\left(2\sqrt{2}\right) = 2\left(\sqrt{2}\right)^2 = 2(2) = 4$

E. $\sqrt{14}\sqrt{2} = \sqrt{28} = \sqrt{4\cdot 7} = \sqrt{4}\sqrt{7} = \mathbf{2}\boldsymbol{\sqrt{7}}$

EXAMPLE 3: **Simplify each radical expression:**

A. $\sqrt{2}\left(1+\sqrt{2}\right) = \sqrt{2}+\sqrt{2}\sqrt{2} = \mathbf{\sqrt{2}+2}$

B. $\sqrt{6}\left(\sqrt{10}-\sqrt{6}\right) = \sqrt{6}\sqrt{10}-\sqrt{6}\sqrt{6} = \sqrt{60}-6 = \mathbf{2\sqrt{15}-6}$

C. $\left(\sqrt{3}+1\right)\left(\sqrt{3}-5\right) = \sqrt{3}\sqrt{3}-5\sqrt{3}+1\sqrt{3}-5$

$= 3-4\sqrt{3}-5 = \mathbf{-2-4\sqrt{3}}$

D. $\left(1+\sqrt{7}\right)^2 = \left(1+\sqrt{7}\right)\left(1+\sqrt{7}\right) = 1+\sqrt{7}+\sqrt{7}+7 = \mathbf{8+2\sqrt{7}}$

E. $\left(3+\sqrt{10}\right)\left(3-\sqrt{10}\right) = 9-3\sqrt{10}+3\sqrt{10}-\sqrt{10}\sqrt{10}$

$= 9-10 = \mathbf{-1}$

Homework

2. Find the product:

a. $\sqrt{14}\sqrt{6}$

b. $\sqrt[3]{3}\times\sqrt[3]{9}$

c. $\sqrt{2}\sqrt{7}$

d. $\sqrt{171}\sqrt{171}$

e. $\sqrt{14}\sqrt{2}$

f. $\sqrt[3]{10}\,\sqrt[3]{12.5}$

g. $\sqrt[4]{2}\,\sqrt[4]{8}$

h. $\sqrt{x}\sqrt{yz}$

i. $\sqrt[3]{2}\,\sqrt[3]{4}$

3. Find the product:

a. $\sqrt{n}\left(1+\sqrt{n}\right)$

b. $\left(x+\sqrt{y}\right)\left(x-\sqrt{y}\right)$

c. $\left(2+\sqrt{3}\right)^2$

d. $\left(\sqrt{p}-q\right)^2$

e. $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$

f. $\left(x-\sqrt{n}\right)^2$

g. $\left(\sqrt{a}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{c}\right)$

h. $2\left(\sqrt{3}+\sqrt{4}\right)$

i. $\left(\sqrt{x}+\sqrt{x}\right)^2$

☐ DIVIDING RADICALS

Just as the square root of a product is the product of the square roots ($\sqrt{ab} = \sqrt{a}\sqrt{b}$), the square root of a quotient is the quotient of the square roots.

THEOREM: Assume a and b are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof: $\sqrt{\dfrac{a}{b}}$

$= \left(\dfrac{a}{b}\right)^{1/2}$ (convert radical to fractional exponent)

$= \dfrac{a^{1/2}}{b^{1/2}}$ (a law of exponents)

$= \dfrac{\sqrt{a}}{\sqrt{b}}$ (convert back to radical form)

Before we get to the heart of dividing radicals, let's recall the little formula $\left(\sqrt[n]{x}\right)^n = x$. Check out the following examples:

$$\left(\sqrt{u}\right)^2 = u \qquad \left(\sqrt[3]{w}\right)^3 = w \qquad \left(\sqrt[4]{xy}\right)^4 = xy$$

$$\sqrt{c}\sqrt{c} = \left(\sqrt{c}\right)^2 = c \qquad \sqrt{17}\times\sqrt{17} = \left(\sqrt{17}\right)^2 = 17$$

Now consider the number $\frac{1}{\sqrt{2}}$. It's just a fraction, and a calculator gives an *approximate* value of 0.707106781. But now consider the scenario where you have no calculator, but you need a decimal version of the fraction. Even if you could look up the positive square root of 2 somehow — and find that it's about 1.414213562 — you'd still be stuck with the following long division problem:

$$1.414213562\,\overline{)\,1.000000000}$$

This is a killer long division problem — it would take a really long time to get just a few digits of the answer.

But now watch this trick: I'm going to take the fraction $\frac{1}{\sqrt{2}}$ and multiply it by the fraction $\frac{\sqrt{2}}{\sqrt{2}}$ (which, of course, equals 1). Here we go:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\left[\frac{\sqrt{2}}{\sqrt{2}}\right]}_{=1} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We've shown that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. [Double-check with your calculator.]

What's the purpose of all this? Consider the long division problem created by the second fraction, $\frac{\sqrt{2}}{2}$:

$$2\,\overline{)\,1.414213562}$$

This is <u>much</u> simpler to carry out than the previous long division problem, and yet it yields the same answer. This is the historical reason why students, especially in the sciences, would convert $\frac{1}{\sqrt{2}}$ into $\frac{\sqrt{2}}{2}$. It made calculations much, much simpler.

Are you going to perform this kind of maneuver — removing radicals from the denominator of a fraction — in your Algebra course? Yes, and for two reasons. One is historical; it's been done like this for centuries. As Tevya from *Fiddler on the Roof* might say, "We do it because it's our tradition!" A second reason: It's a lot easier when we look up answers in the "back of the book" if we can all agree on what form to leave our answers in.

I'll let you in on a little secret: If you take Calculus, there will be problems where you will <u>want</u> to leave radicals in the denominator. Indeed, you will actually remove radicals from the <u>numerator</u>.

We thus have a basic agreement in math: Do not leave radicals in the denominator of a fraction. Converting a fraction with a radical in the denominator into an equivalent fraction without a radical in the denominator is called ***rationalizing the denominator***.

Some trig books will leave answers like $\frac{1}{\sqrt{3}}$, so be prepared.

EXAMPLE 4:

A. Rationalize the denominator, which is a **monomial**:

$$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}}\underbrace{\left[\frac{\sqrt{x}}{\sqrt{x}}\right]}_{=1} = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

B. Rationalize the denominator, which is a **binomial**:

$$\frac{\sqrt{2}}{5-\sqrt{10}}$$

For this example, the trick is in figuring out what, exactly, we should multiply the top and the bottom of the fraction by.

$$\frac{\sqrt{2}}{5-\sqrt{10}} = \frac{\sqrt{2}}{5-\sqrt{10}}\underbrace{\left[\frac{5+\sqrt{10}}{5+\sqrt{10}}\right]}_{=1} = \frac{5\sqrt{2}+\sqrt{20}}{25-10} = \frac{5\sqrt{2}+2\sqrt{5}}{15}$$

Notes: Many students try multiplying top and bottom by $\sqrt{10}$, figuring that it would remove the $\sqrt{10}$. Try it and you'll see this procedure fail.

Other students think they're more clever and multiply top and bottom by $5-\sqrt{10}$. Go for it; you'll still have radicals in the bottom.

The quantity $5+\sqrt{10}$, which does eradicate the radical from the bottom, is called the ***conjugate*** of $5-\sqrt{10}$.

Homework

4. Simplify so that there are no radicals in the denominator:

a. $\frac{2}{\sqrt{8}}$ b. $\frac{x^2}{\sqrt{x}}$ c. $\frac{3}{5+\sqrt{2}}$ d. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\frac{1}{\sqrt{3}}$ f. $\frac{6}{\sqrt{9}}$ g. $\frac{8}{\sqrt{2}}$ h. $\frac{\sqrt{x}}{\sqrt{y}}$

i. $\frac{5}{1+\sqrt{5}}$ j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ k. $\frac{7}{3\sqrt{2}}$ l. $\frac{x}{n\sqrt{x}}$

m. $\frac{7}{1+\sqrt{7}}$ n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ o. $\frac{a}{b\sqrt{c}}$ p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$

5. Two of the following are valid, while two of them are bogus. Which are which?

a. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ b. $\sqrt{a-b} = \sqrt{a}-\sqrt{b}$

c. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ d. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Review Problems

6. a. Find the two fourth roots of 256.

b. Find all the cube roots of –1000.

c. Find all the square roots of –121.

7. Simplify each expression:

a. $\sqrt{7}\sqrt{5}$ b. $\sqrt{7}+\sqrt{5}$ c. $\dfrac{\sqrt{7}}{\sqrt{5}}$ d. $\sqrt{7}-\sqrt{5}$

e. $\sqrt{2}+\sqrt[3]{2}$ f. $\sqrt{28}+\sqrt{175}$ g. $\left(\sqrt{h}+7\right)^2$ h. $\left(3-\sqrt{t}\right)^2$

8. Simplify so that there are no radicals in the denominator:

a. $\dfrac{4}{\sqrt{35}-5}$ b. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ c. $\dfrac{12}{\sqrt{9}+3}$

9. Simplify each expression:

a. $\sqrt{18}\sqrt{8}$ b. $\sqrt{7}-\sqrt{7}$ c. $\dfrac{1+\sqrt{7}}{\sqrt{7}}$ d. $\sqrt{50}-\sqrt{8}$

e. $\dfrac{3}{\sqrt{28}+5}$ f. $\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$ g. $\dfrac{3-\sqrt{25}}{\sqrt{25}-3}$

10. Let $r_1 = \dfrac{3+\sqrt{11}}{4}$ and $r_2 = \dfrac{3-\sqrt{11}}{4}$.

Prove a. $r_1+r_2 = \dfrac{3}{2}$ b. $r_1 \times r_2 = -\dfrac{1}{8}$

11. True/False:

a. 36 has two square roots.
b. $\sqrt{49} = \pm 7$
c. If $x \le 0$, then $\sqrt{x}$ is not a real number.
d. 64 has exactly one cube root.
e. $-\sqrt[4]{20}$ is a real number.
f. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ Assume $a, b \ge 0$
g. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ Assume $a, b \ge 0$
h. $\sqrt[3]{18} = 3\sqrt{2}$
i. $\sqrt{20}+\sqrt{5} = 5$
j. $\sqrt{20}+\sqrt{5} = 3\sqrt{5}$

> *"The wisest mind has something yet to learn."*
>
> **George Santayana (1863 - 1952)**

k. $\left(\sqrt{a}+\sqrt{b}\right)^2 \;=\; a+b+2\sqrt{ab}$

l. $\dfrac{7}{\sqrt{n}} \;=\; 7\sqrt{n}$

m. $\dfrac{1}{\sqrt{n}+1} \;=\; \dfrac{\sqrt{n}-1}{n-1}$

n. $\sqrt[3]{x^2}+\sqrt[3]{x} \;=\; x$

☐ *TO ∞ AND BEYOND*

A. Show that $\dfrac{\sqrt{6}+\sqrt{2}}{4} \;=\; \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

Note: A calculator gives good evidence, but it's not a proof.

B. Show that $2\dfrac{r}{\sqrt{2}}\sqrt{r^2-\dfrac{r^2}{2}} \;=\; r^2$

Solutions

1. a. $12\sqrt{x}$ b. As is c. $13\sqrt[3]{n}$ d. As is e. $68\sqrt{2}$
f. $5\sqrt[3]{3}$ g. $-3\sqrt{5}$ h. $32\sqrt[3]{7}$ i. $3\sqrt[3]{z^2}$ j. $12\sqrt{3}$
k. $10\sqrt{5}$ l. $\sqrt{11}$

2. a. $\sqrt{84} \;=\; 2\sqrt{21}$ b. $\sqrt[3]{27} \;=\; 3$ c. $\sqrt{14}$
d. 171 e. $\sqrt{28} \;=\; 2\sqrt{7}$ f. $\sqrt[3]{125} \;=\; 5$
g. $\sqrt[4]{16} \;=\; 2$ h. $\sqrt{xyz}$ i. $\sqrt[3]{8} \;=\; 2$

3. a. $\sqrt{n}+n$ b. $x^2-x\sqrt{y}+x\sqrt{y}-y \;=\; x^2-y$
c. $4+4\sqrt{3}+3 \;=\; 7+4\sqrt{3}$ d. $\left(\sqrt{p}-q\right)\left(\sqrt{p}-q\right) \;=\; p-2q\sqrt{p}+q^2$
e. 7 – 4 = 3 f. $x^2-2x\sqrt{n}+n$ g. $a-c$

h. $2\left(\sqrt{3}+2\right) = 2\sqrt{3}+4$ i. $\left(2\sqrt{x}\right)^2 = 4x$

4. a. $\frac{2}{\sqrt{8}}\left[\frac{\sqrt{8}}{\sqrt{8}}\right] = \frac{2\sqrt{8}}{8} = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2}$ b. $x\sqrt{x}$

c. $\frac{3}{5+\sqrt{2}}\left[\frac{5-\sqrt{2}}{5-\sqrt{2}}\right] = \frac{15-3\sqrt{2}}{25-2} = \frac{15-3\sqrt{2}}{23}$ d. $\frac{x+2\sqrt{xy}+y}{x-y}$

e. $\frac{\sqrt{3}}{3}$ f. 2 g. $4\sqrt{2}$ h. $\frac{\sqrt{xy}}{y}$

i. $\frac{5}{1+\sqrt{5}}\left[\frac{1-\sqrt{5}}{1-\sqrt{5}}\right] = \frac{5-5\sqrt{5}}{1-5} = \frac{5-5\sqrt{5}}{-4} = \frac{5-5\sqrt{5}}{-4}\left[\frac{-1}{-1}\right] = \frac{5\sqrt{5}-5}{4}$

The last step is optional. It was done just to remove the negative sign from the bottom of the fraction.

j. $-3-2\sqrt{2}$ k. $\frac{7}{3\sqrt{2}}\left[\frac{\sqrt{2}}{\sqrt{2}}\right] = \frac{7\sqrt{2}}{3\cdot 2} = \frac{7\sqrt{2}}{6}$ l. $\frac{\sqrt{x}}{n}$

m. $\frac{7\sqrt{7}-7}{6}$ n. $\frac{a^2+2a\sqrt{b}+b}{a^2-b}$ o. $\frac{a\sqrt{c}}{bc}$ p. 1

5. c. and d. are valid — a. and b. are bogus.

6. a. ± 4 b. –10 c. None

7. a. $\sqrt{35}$ b. As is c. $\frac{\sqrt{35}}{5}$ d. As is

e. As is f. $7\sqrt{7}$ g. $h+14\sqrt{h}+49$ h. $9-6\sqrt{t}+t$

8. a. $\frac{2\sqrt{35}+10}{5}$ b. $\frac{a-2\sqrt{ab}+b}{a-b}$ c. 2

9. a. 12 b. 0 c. $\frac{\sqrt{7}+7}{7}$ d. $3\sqrt{2}$

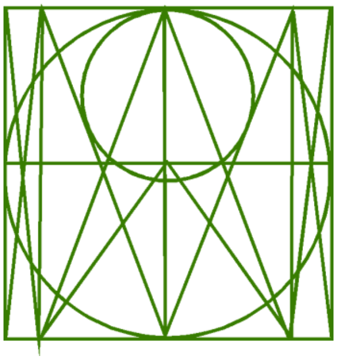
e. $2\sqrt{7}-5$ f. $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$ g. –1

10. Do the arithmetic and it should work out.

11. a. T b. F c. F d. T e. T f. F g. T h. F

i. F j. T k. T l. F m. T n. F

CH NN – RATIONAL FUNCTIONS



Remember what we call a number like $\frac{2}{7}$? This is called a *rational number* because it is the *ratio* of two integers. In a like manner, a *rational function* is the ratio of two special functions called ***polynomial functions***. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain of a such a function may not be all real numbers.

POLYNOMIAL FUNCTIONS

Each of the following is a polynomial function:

$y = 7$	(a *linear* function – it's a horizontal line)
$y = -3x + \sqrt{2}$	(a *linear* function – it's a line with slope = –3)
$y = 2x^2 - x + 9$	(a *quadratic* function – it's a parabola)
$f(t) = \sqrt[4]{2}\,t^3 - t^2$	(a *cubic* function)
$P(w) = -\pi w^4 + 5w^2 + 8$	(a *quartic* function)
$Q(a) = \frac{2}{3}a^5 - 4a + 1$	(a *quintic* function)

The key to any ***polynomial function*** is that all the exponents on the input variable come from the set of whole numbers: $\{0, 1, 2, 3, \ldots\}$. The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in $\mathbb{R}$, the set of real numbers.

Consider the quartic (4th degree) polynomial function

$$y = -2\pi x^4 - \tfrac{9}{10}x^3 + 9.2x + \sqrt{2}$$

First look at the exponents; they are all whole numbers. Notice that the x in the term $9.2x$ has an understood exponent of 1. Even the last term, $\sqrt{2}$, can be written as $\sqrt{2}\,x^0$, and so even the exponent on this last term is a whole number.

Thus, all the exponents on the x's (4, 3, 1, and 0) come from the whole numbers, while all the coefficients (-2π, $-\tfrac{9}{10}$, 9.2, $\sqrt{2}$) come from $\mathbb{R}$. Considering the definition of polynomial function, the given function is indeed a polynomial function.

Each of the following is <u>not</u> a polynomial function:

$y = \dfrac{1}{x}$	($\tfrac{1}{x} = x^{-1}$ and -1 is not a whole number)
$y = \sqrt{x}$	($\sqrt{x} = x^{1/2}$ and $\tfrac{1}{2}$ is not a whole number)
$f(x) = \dfrac{1}{\sqrt[3]{x}}$	($\tfrac{1}{\sqrt[3]{x}} = x^{-1/3}$ and $-\tfrac{1}{3}$ is not a whole number)
$g(x) = \lvert x-1 \rvert$	(no absolute values allowed around the x)
$E(x) = 2^x$	(since x is in the exponent, it can be any number)
$T(x) = \sin x$	(it's on your calculator, but it's not a polynomial function)
$y = \log x$	(a log function can never be a polynomial function)
$x^2 + y^2 = 25$	(it's a circle – it's not a function of any kind)

Homework

1. Explain why $y = \pi x^5 - \sqrt{2}\,x^3 + \tfrac{1}{4}x - 17.5$ <u>is</u> a polynomial function.

2. Explain why $h(x) = \sqrt[3]{5}x^4 - \sqrt{2x} + \frac{1}{2}$ is <u>not</u> a polynomial function.

3. The highest exponent on the variable in a polynomial function is called its ***degree***. Find the degree of each polynomial function:

   a. $y = \pi$ b. $y = x^3 - 17x^2 + 8$

   c. $y = \sqrt{2}x^8 - \frac{\pi}{2}x^{10}$ d. $y = 7x + 5$

4. a. What is the domain of any polynomial function?

   b. T/F: Every parabola is a polynomial function.

   c. T/F: Every non-vertical line is a polynomial function.

   d. T/F: Every line is a polynomial function.

❑ *RATIONAL FUNCTIONS*

A ***rational function*** is defined to be the *ratio* of two *polynomial functions*. If P and Q are polynomial functions, then $R = \frac{P}{Q}$ is a rational function. A typical example of a rational function is $y = \frac{3.9x^2 + 7x - 9}{x + 8}$ (a quadratic function over a linear function).

<u>EXAMPLE 1:</u> **Graph:** $\boldsymbol{y = \frac{1}{x - 2}}$

<u>Solution:</u> Since y is the ratio of a constant polynomial function and a linear polynomial function, we know that y is a rational function.

We begin our analysis of this rational function by determining the **domain**. In order that the fraction be defined, we must <u>not</u> divide by zero. What value of x makes the denominator zero? The value $x = 2$ will. (Just set $x - 2 = 0$ and solve for x.)

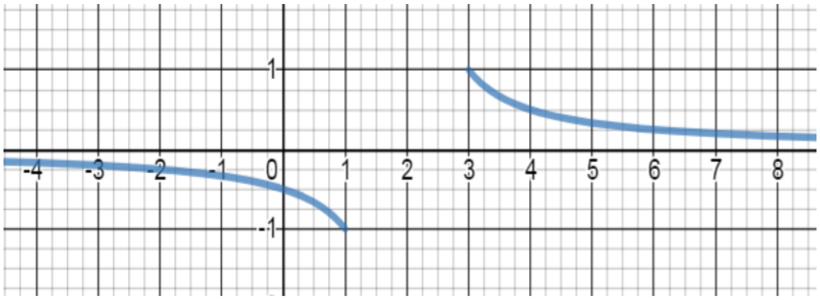
Therefore, the domain is all real numbers except 2; that is, the domain is **ℝ – {2}**.

Intercepts come next. If $x = 0$, then $y = \frac{1}{0-2} = -\frac{1}{2}$. Thus, $\mathbf{(0, -\frac{1}{2})}$ is the y-intercept. To find an x intercept, set $y = 0$. This gives $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$, which has no solution. Thus, there are **no** x-intercepts.

Now for some ordered pairs that satisfy the formula $y = \frac{1}{x-2}$:

x	y
–3	$-\frac{1}{5}$
–2	$-\frac{1}{4}$
–1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	–1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$

If we plot these points and connect them with a smooth curve, we would get the following graph:

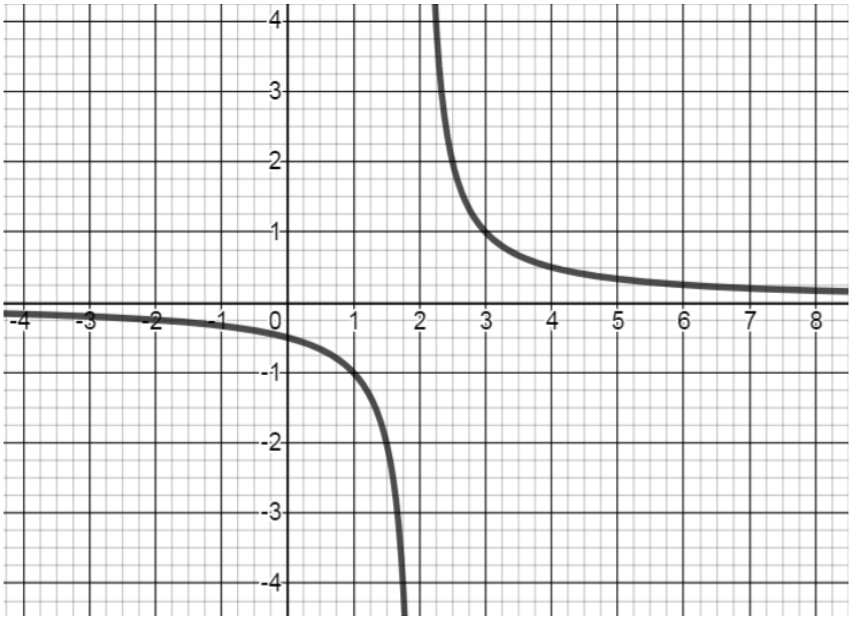


What some students do at this point is to lazily connect the points (3, 1) and (1, –1) with a straight line. Talk about jumping to conclusions! Our domain of ℝ – {2} implies that x <u>cannot</u> be 2 in this function; the straight-line trick won't work. So we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some x-values that are <u>near</u> 2:

$(1\frac{1}{2}, -2)$ $(1\frac{3}{4}, -4)$ $(1\frac{7}{8}, -8)$

$(2\frac{1}{2}, 2)$ $(2\frac{1}{4}, 4)$ $(2\frac{1}{8}, 8)$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some real cool **limits**. Suppose we let x approach ∞. The y-values are positive (the curve is above the x-axis), but are getting smaller and smaller, approaching zero. Thus, **as $x \to \infty, y \to 0$**. [This can be read: "*As x grows infinitely large, y is getting closer and closer to 0.*]

Now let x approach –∞. The y-values are negative but are rising toward zero. Therefore, as $x \to -\infty, y \to 0$.

The number 2 seems to be an interesting x-value. Although x can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line $x = 2$. In fact, if we let x approach 2 from the right, the curve is growing taller and taller, and so we have the limit: **As $x \to 2$ (from the right), $y \to \infty$**. [This can be read: "*As x gets closer and closer to 2, approaching 2 from the right (meaning values larger than 2), y is growing infinitely large.*]

Now let x approach 2 from the left. This time the curve is dropping rapidly, toward negative infinity. This observation yields the limit: **As $x \to 2$ (from the left), $y \to -\infty$.**

Let's summarize the four limits we've deduced:

- As $x \to \infty$, $y \to 0$.
- As $x \to -\infty$, $y \to 0$.
- As $x \to 2$ (from the right), $y \to \infty$.
- As $x \to 2$ (from the left), $y \to -\infty$.

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the x-axis? We say that the line $\boldsymbol{y = 0}$ (which is the x-axis) is a ***horizontal asymptote***.

Now look at the region of the graph near $x = 2$. The curve gets closer and closer to the vertical line $x = 2$ (in fact, on both sides of the vertical line). We call the line $\boldsymbol{x = 2}$ a ***vertical asymptote***.

EXAMPLE 2: **Graph:** $y = \dfrac{2x-1}{x+2}$

Solution: First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when $x = -2$. Thus, the domain is $\mathbb{R} - \{-2\}$.

Now let's explore the **intercepts**:

If $x = 0$, then $y = \dfrac{2(0)-1}{0+2} = -\dfrac{1}{2}$. There's a y-intercept at $\mathbf{(0, -\tfrac{1}{2})}$.

If $y = 0$, then $0 = \dfrac{2x-1}{x+2} \Rightarrow 2x-1 = 0 \Rightarrow x = \tfrac{1}{2}$. So $\mathbf{(\tfrac{1}{2}, 0)}$ is an x-intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

(–1, –3)	(1, 0.33)	(3, 1)	(5, 1.29)	(10, 1.58)

(15, 1.71)   (20, 1.77)   (100, 1.95)   (1000, 1.995)

What's happening as x grows very large? It appears that y is approaching 2. That is, **as $x \to \infty$, $y \to 2$**.

Now we'll let x go the other direction:

(–3, 7)   (–5, 3.67)   (–10, 2.63)   (–20, 2.28)

(–100, 2.05)   (–1000, 2.01)

These points show that as **$x \to -\infty$, $y \to 2$**.

Finally, here are some ordered pairs for x's near –2 (the only real number not in the domain):

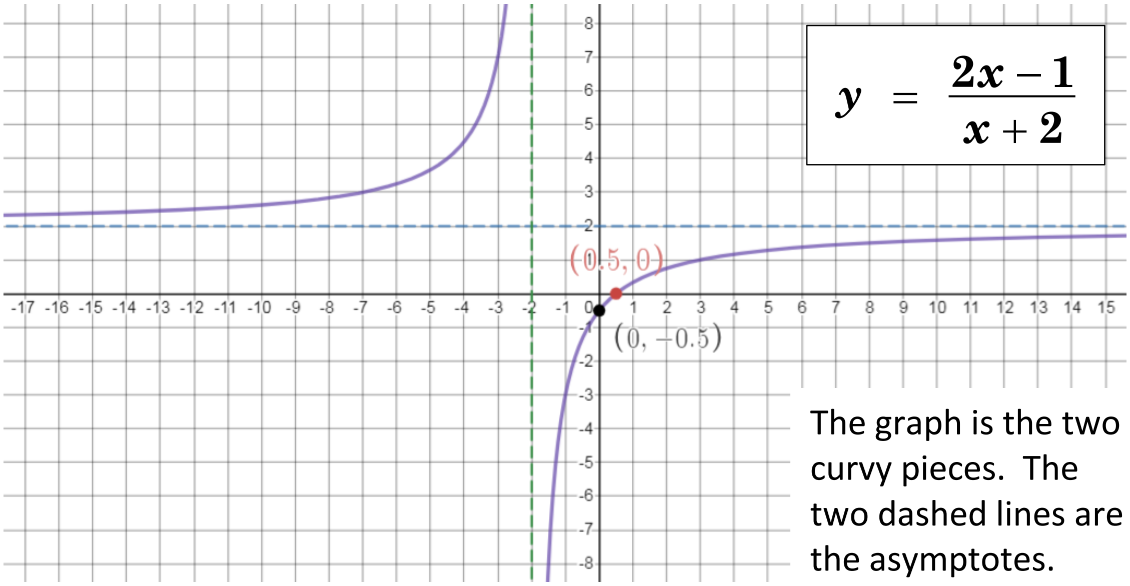
(–1.5, –8)   (–1.9, –48)   (–1.99, –498)

Thus, **as $x \to -2$ (from the right), $y \to -\infty$**.

(–2.5, 12)   (–2.1, 52)   (–2.01, 502)

So, as $x \to -2$ (from the left), $y \to \infty$.

Plotting as many of the calculated points as possible, including the two intercepts, and considering the four limits we've found, the following graph emerges:



The graph is the two curvy pieces. The two dashed lines are the asymptotes.

We can now be reasonably sure of the **asymptotes** (denoted by the dashed lines). Either by recalling the limits described above or by looking at the graph, we conclude that there's **a vertical asymptote at $x = -2$** and **a horizontal asymptote at $y = 2$**.

EXAMPLE 3: **Graph:** $\boldsymbol{y = \dfrac{4}{1+x^2}}$

Solution: Why is this function rational? Because it's the *ratio* $\dfrac{P}{Q}$ of two polynomial functions: the constant polynomial $P(x) = 4$ and the quadratic polynomial $Q(x) = 1 + x^2$.

To find the **domain**, set the denominator to zero to see what's not in the domain: $1 + x^2 = 0$. This equation has no solution in $\mathbb{R}$, since solving it leads to $x = \pm\sqrt{-1}$, which are not real numbers. In fact, for any value of x, the quantity $1 + x^2$ is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is $\mathbb{R}$. We can also figure that the graph will not have a **vertical asymptote**, since the denominator can never be 0.

Notice that if we put in some positive value of x, we'll get a certain y-value. Now look at what will happen if we put $-x$ (the *opposite* of x) into the formula. Since $(-x)^2$ is equal to x^2, we will get the same y-value. This implies that the left side of the graph will be the mirror image of the right side. We say that the graph has ***y*-axis symmetry** (or is *symmetric with respect to the y-axis*).

Now we seek the **intercepts**. Set $x = 0$ to get $y = 4$, and so the y-intercept is **(0, 4)**. Now set $y = 0$, giving

$$0 = \frac{4}{1+x^2} \Rightarrow 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \Rightarrow 0 = 4$$

This absurd result indicates that the equation has no solution; hence, there are **no** x-intercepts.

It's time for some additional ordered pairs for this function:

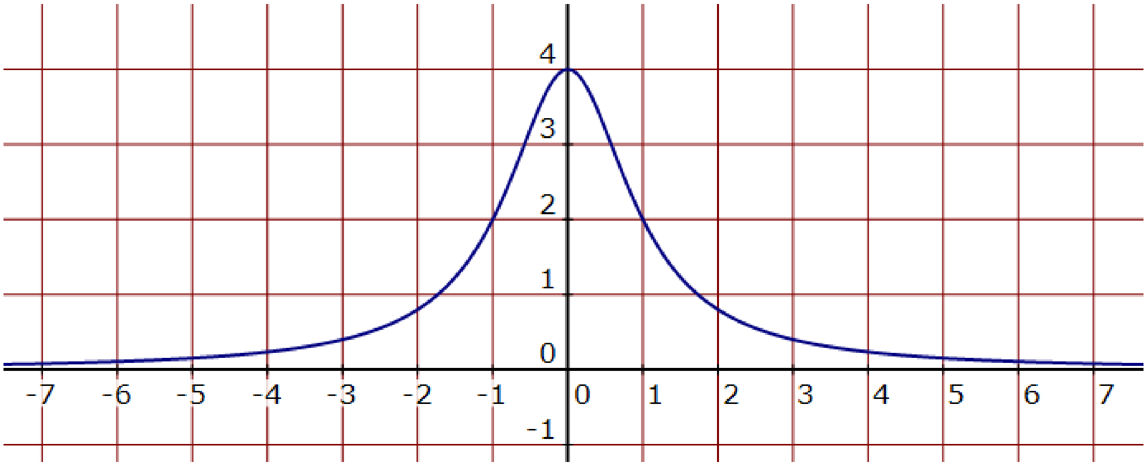
(1, 2) (2, 0.8) (3, 0.4) (4, 0.24) (10, 0.04) (200, 0.0001)

These points suggest the limit: **As $x \to \infty$, $y \to 0$**. This implies that **$y = 0$ is a horizontal asymptote**.

Here are some more ordered pairs, designed to see what happens as we approach the *y*-axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

If we plot all the points calculated so far, and if we recall the *y*-axis symmetry, we get the following graph:



We determined at the outset that the *domain* of this rational function is ℝ. Is it clear from the graph that this is indeed the case?

If you happen to know what the **range** of a function is, you can see that the range of this function is all real numbers that are greater than 0 but less than or equal to 4: **Range = (0, 4]**

Homework

5. Consider the rational function in Example 2: $y = \frac{2x-1}{x+2}$.

   Without referring to the graph, prove that y can have the value 2.01, but y can never have the value 2.

6. Find the domain:

   a. $y = \frac{2x+7}{9}$ b. $f(x) = \frac{2x-3}{4+x}$

   c. $g(x) = \frac{3x}{2x-10}$ d. $R(x) = \frac{x-1}{-7x+4}$

   e. $f(x) = \frac{x^2-9}{x^2-100}$ f. $g(x) = \frac{8x-16}{x^2+25}$

7. Find the intercepts:

   a. $f(x) = \frac{x-4}{x-2}$ b. $y = \frac{3}{5x-15}$

   c. $y = \frac{2x+1}{x-3}$ d. $g(x) = \frac{5-x}{6x+1}$

8. Find the asymptotes:

   a. $R(x) = \frac{8x+1}{4x-4}$ b. $y = \frac{2x-3}{2x+1}$

   c. $y = \frac{3x-7}{x+2}$ d. $h(x) = \frac{2x+7}{4x-4}$

9. Find the domain, the intercepts, and the asymptotes of

$$y = \frac{1}{4+x^2}$$

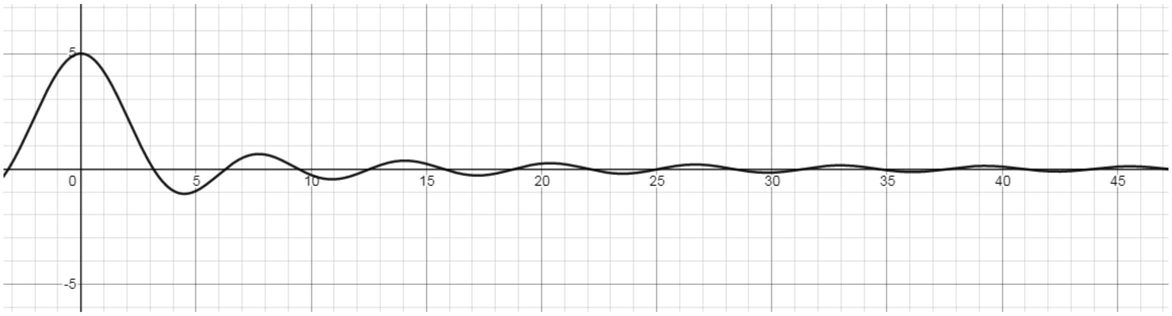
10. Perform a complete analysis of the function $y = \frac{2}{x-3}$.

11. Perform a complete analysis of the function $y = \dfrac{3x-5}{x-2}$.

12. Perform a complete analysis of the function $y = \dfrac{2}{2+x^2}$.

13. Perform a complete analysis of the function $y = \dfrac{-1}{x+1}$.

14. Consider the graph



Explain why the horizontal line $y = 0$ (that is, the x-axis) is a horizontal asymptote for the curve.

Review Problems

15. a. Explain why $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$ is a polynomial function. What is its degree?

b. Explain why $y = 3x^5 - \sqrt{x} + \pi$ is not a polynomial function.

16. a. A horizontal line (is, is not) a polynomial function.
b. The function $y = \dfrac{1}{x}$ (is, is not) a polynomial function.
c. What is the degree of the polynomial function $y = 7x - \pi$?
d. Is a circle a polynomial function?

17. Consider the rational function $y = \dfrac{7}{2x-8}$.

    a. Find the domain.
    b. Find all the intercepts.
    c. Find all the asymptotes.
    d. Calculate y if $x = 4.1$.

18. Find all the intercepts and asymptotes of $r(x) = \dfrac{8x+6}{2x-3}$, and graph.

19. Graph $y = \dfrac{-2x-2}{x-1}$.

As $x \to 1$ (from the right), $y \to$ ___.
As $x \to 1$ (from the left), $y \to$ ___.
As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.

20. Graph $y = \dfrac{5}{2+x^2}$. Discuss domain, symmetry, and asymptotes.

As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.
As $x \to 0$ (from the right), $y \to$ ___.
As $x \to 0$ (from the left), $y \to$ ___.

21. True/False:

a. $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$ is a polynomial function.

b. $y = \dfrac{1}{x^5}$ is a polynomial function.

c. The graph of $f(x) = \dfrac{1}{2x+10}$ has a vertical asymptote at $x = -5$.

d. The graph of $g(x) = \dfrac{10x+9}{5x-11}$ has a horizontal asymptote at $y = 10$.

e. The domain of the function $y = \dfrac{6}{1+x^2}$ is $\mathbb{R} - \{\pm 1\}$.

f. For the graph of $y = \frac{3x+1}{x-\pi}$, as $x \to \infty$, $y \to 3$.

Solutions

1. All coefficients are from $\mathbb{R}$, and all exponents are from $\mathbb{W}$ (the whole numbers).

2. The middle term is $\sqrt{2}\,x^{1/2}$, and $\frac{1}{2} \notin \mathbb{W}$.

3. a. 0 b. 3 c. 10 d. 1

4. a. $\mathbb{R}$ b. False c. True d. False

5. $y = \frac{2x-1}{x+2} \Rightarrow 2.01 = \frac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502$.
So, (–502, 2.01) is on the graph, and indeed y can be 2.01.

Now let's pretend that y could be 2; then
$2 = \frac{2x-1}{x+2} \Rightarrow 2x + 4 = 2x - 1 \Rightarrow 4 = -1 \Rightarrow$ No solution. Thus, there is no x which will make $y = 2$.

6. a. $\mathbb{R}$ b. $\mathbb{R} - \{-4\}$ c. $\mathbb{R} - \{5\}$ d. $\mathbb{R} - \left\{\frac{4}{7}\right\}$ e. $\mathbb{R} - \{\pm 10\}$ f. $\mathbb{R}$

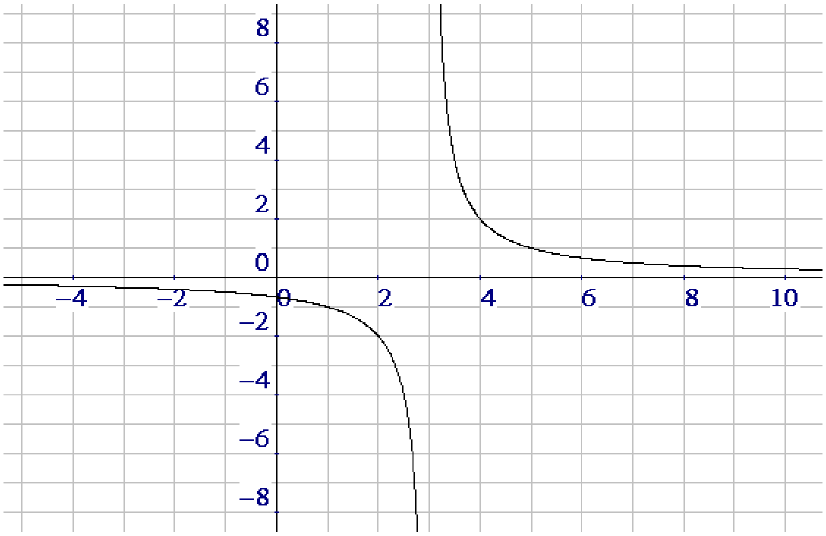
7. a. (4, 0) (0, 2) b. $(0, -\frac{1}{5})$ c. $(-\frac{1}{2}, 0)$ $(0, -\frac{1}{3})$ d. (5, 0) (0, 5)

8. a. $x = 1$ $y = 2$ b. $x = -\frac{1}{2}$ $y = 1$ c. $x = -2$ $y = 3$ d. $x = 1$ $y = \frac{1}{2}$

9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is $\mathbb{R}$.

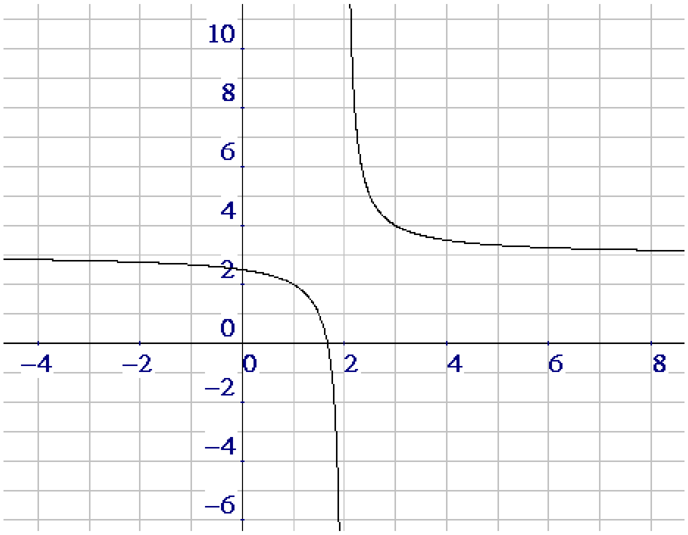
Setting $x = 0$ gives a y-value of 1/4, so the y-intercept is $(0, \frac{1}{4})$. If you set $y = 0$, you'll get no solution for y. Thus, there is no x-intercept.
There are no vertical asymptotes, since the denominator is never zero.
Letting x approach either ∞ or $-\infty$, y approaches 0. Thus, a horizontal asymptote is $y = 0$ (the x-axis).

10.



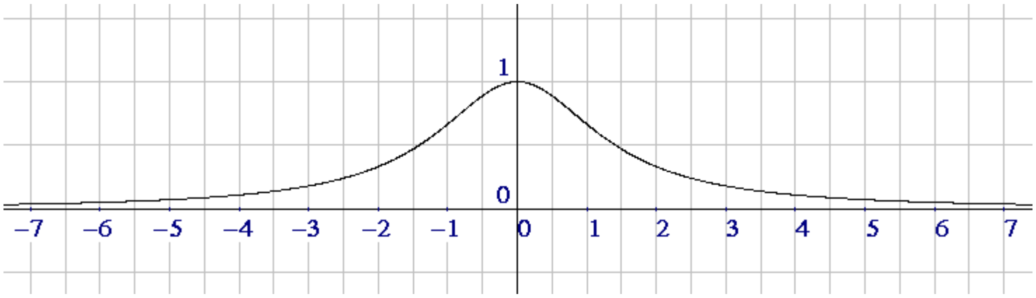
Domain = $\mathbb{R} - \{3\}$
x-int: none
y-int: $(0, -\frac{2}{3})$
As $x \to 3$ (from the right), $y \to \infty$
As $x \to 3$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$
vert asy: $x = 3$
horiz asy: $y = 0$

11.



Domain = $\mathbb{R} - \{2\}$
x-int: $(\frac{5}{3}, 0)$
y-int: $(0, \frac{5}{2})$
As $x \to 2$ (from the right), $y \to \infty$
As $x \to 2$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 3$
As $x \to -\infty$, $y \to 3$
vert asy: $x = 2$
horiz asy: $y = 3$

12.



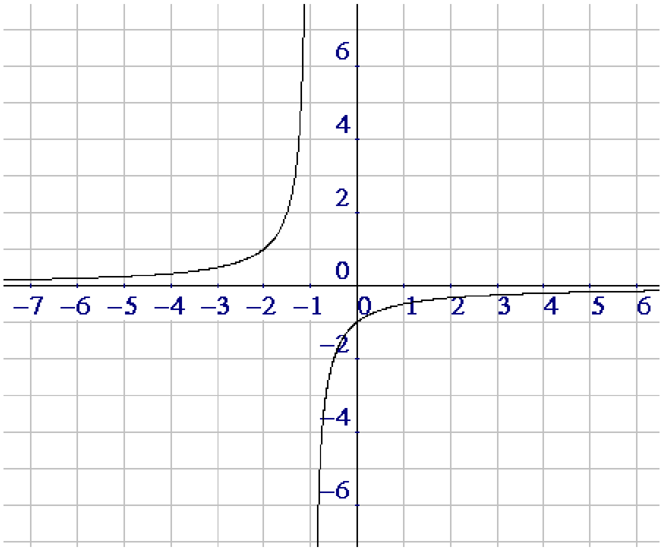
Domain = $\mathbb{R}$
x-int: none
y-int: (0, 1)

As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$

vert asy: none
horiz asy: $y = 0$

maximum point at (0, 1)

13.



Domain = $\mathbb{R} - \{-1\}$

x-int: none

y-int: (0, −1)

As $x \to -1$ (from the right), $y \to -\infty$

As $x \to -1$ (from the left), $y \to \infty$

As $x \to \infty$, $y \to 0$

As $x \to -\infty$, $y \to 0$

vert asy: $x = -1$

horiz asy: $y = 0$

14. Because of the limit: As $x \to \infty$, $y \to 0$. Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the x-axis (the line $y = 0$), and this is ultimately what is meant by a horizontal asymptote.

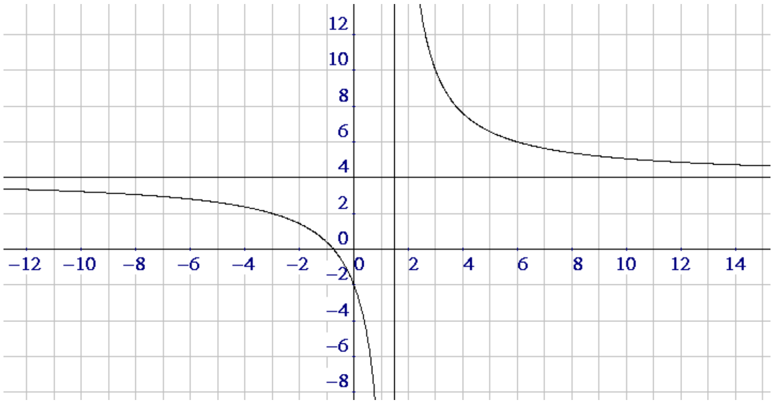
15. a. f is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.

b. Look at the middle term; it can be written as $x^{1/2}$, a term whose exponent is not from the whole numbers.

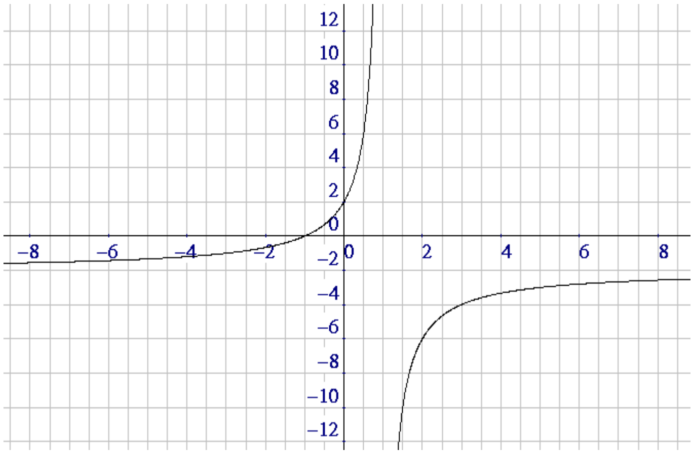
16. a. is b. is not ($1/x = x^{-1}$) c. 1 d. It's not even a function, let alone the special function called a polynomial.

17. a. $\mathbb{R} - \{4\}$ b. $(0, -7/8)$ c. $x = 4$ and $y = 0$ d. 35

18. Intercepts: $(0, -2)$ and $(-\frac{3}{4}, 0)$; vert asy: $x = \frac{3}{2}$; horiz asy: $y = 4$

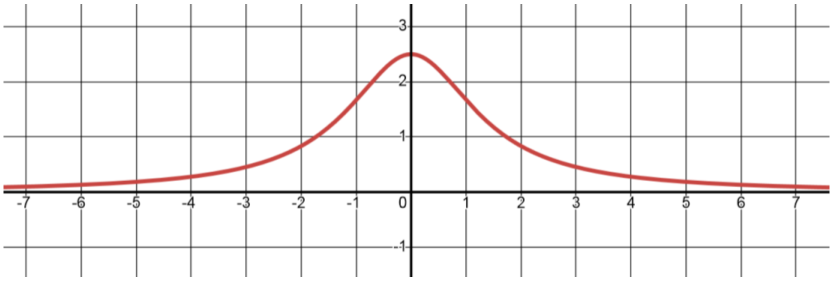


19.



Limits: $-\infty$; ∞; -2; -2

20.



Domain = $\mathbb{R}$

y-axis symmetry

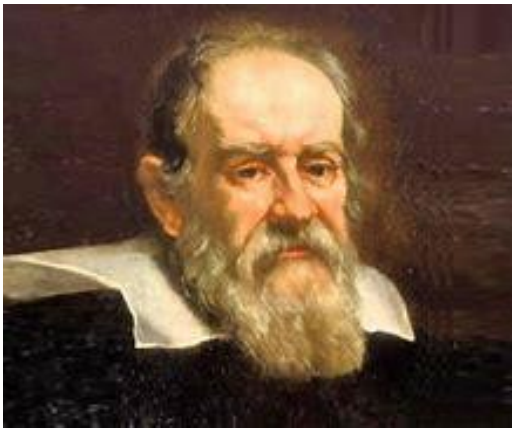
No vert asy

Horiz asy: $y = 0$

Limits: 0; 0; $\frac{5}{2}$; $\frac{5}{2}$

21. a. T b. F c. T d. F e. F f. T

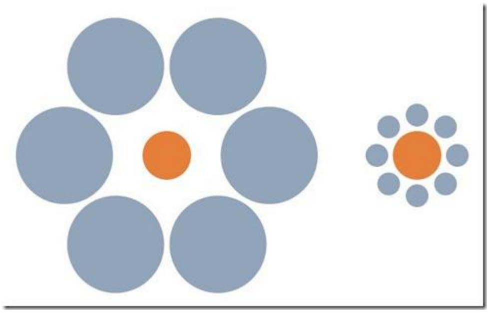
“ *The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language.*”



Galileo Galiliei

RATIOS AND PROPORTIONS

❑ INTRODUCTION



ratio comparison by division

A ***ratio*** is basically a fraction, and a ***proportion*** is a statement that two ratios are equal. An example of a proportion is

$$\frac{16}{12} = \frac{8}{6}$$

These ratios are equal because both fractions could be reduced to $\frac{4}{3}$ — or, if you prefer, they both equal 1.333....

❑ SOLVING SIMPLE PROPORTIONS

common sense: "raising and reducing fractions.

Example 1: **by solving the equation.**

$$\frac{5}{13} = \frac{x}{1287}$$

will be a proportion (two equal ratios) as soon as we find the value of x which will make the equation true. To solve this proportion, we can isolate the x by multiplying each side of the equation by 1287. It might be useful to write 287 as a fraction with denominator 1:

$$\frac{5}{13}\left[\frac{1287}{1}\right] = \frac{x}{\cancel{1287}}\left[\frac{\cancel{1287}}{1}\right]$$

This simplifies to

$$\frac{6435}{13} = x\text{, or } \underline{\boldsymbol{x = 495}}$$

<u>Example 2:</u> To solve the proportion

$$\frac{18}{n} = \frac{30}{17}$$

we notice that the variable n is on the bottom of the first ratio — we have to get it to the top so that we can solve for it. So first we'll multiply each side of the equation by n:

$$\frac{18}{n}\left[n\right] = \frac{30}{17}\left[n\right]$$

As before, we'll write each of the n's that we've put into the problem as a fraction with denominator 1. Thus, we can write

$$\frac{18}{\cancel{n}}\left[\frac{\cancel{n}}{1}\right] = \frac{30}{17}\left[\frac{n}{1}\right]$$

which simplifies to

$$18 = \frac{30n}{17}$$

Now multiply each side of the equation by 17 (or $\frac{17}{1}$ if you'd prefer):

$$18\left[17\right] = \frac{30n}{\cancel{17}}\left[\cancel{17}\right],$$

which (after flipping around the equal sign) leads to the equation

$$30n = 306$$

Now divide each side by 30:

$$\frac{30n}{30} = \frac{306}{30}$$

and we have our solution: $\underline{\boldsymbol{n = 10.2}}$

Homework

1. Which of the following statements are true proportions? [Try reducing fractions or converting fractions to decimals.]

a. $\frac{60}{120} = \frac{1}{2}$ b. $\frac{37}{111} = \frac{10}{30}$ c. $\frac{4}{5} = \frac{5}{6}$

d. $\frac{1}{7} = \frac{2}{13}$ e. $\frac{336}{84} = \frac{68}{17}$ f. $\frac{14}{6} = \frac{7}{2}$

2. Solve each proportion (leave answers rounded to 2 digits):

a. $\frac{x}{7} = \frac{2}{9}$ b. $\frac{14}{a} = \frac{1}{20}$ c. $\frac{2}{7.2} = \frac{b}{5}$

d. $\frac{100}{9.1} = \frac{2.3}{u}$ e. $\frac{c}{2.3} = \frac{1}{1.1}$ f. $\frac{14}{y} = \frac{14}{37}$

3. Solve each proportion (leave answers rounded to 2 digits):

a. $\frac{a}{13} = \frac{14}{156}$ b. $\frac{23}{b} = \frac{1}{14}$ c. $\frac{3}{4} = \frac{n}{128}$

d. $\frac{9}{2} = \frac{7.8}{z}$ e. $\frac{13}{u} = \frac{23}{10}$ f. $\frac{17}{15} = \frac{0.23}{k}$

g. $\frac{x}{5} = -7$ h. $\frac{17}{n} = -\frac{4}{5}$ i. $\frac{a}{9} = \frac{9}{a}$ [2 answers]

j. $\frac{x}{3} = \frac{1}{7}$ k. $\frac{2.1}{y} = \frac{5}{7}$ l. $\frac{8}{9} = \frac{w}{0.8}$

m. $\frac{2.1}{5} = \frac{9.3}{z}$ n. $\frac{5.6}{a} = \frac{2.33}{0.04}$ o. $\frac{x}{2} = \frac{18}{x}$ [2 answers]

❑ SOLVING MORE PROPORTIONS

EXAMPLE 3: **Solve for x:** $\mathbf{\frac{3x-9}{5} = \frac{2x+7}{4}}$

Solution: This proportion looks more complicated than the previous ones, but you know what? We have all the tools we need to solve it. Let's begin by writing the original problem:

$$\frac{3x-9}{5} = \frac{2x+7}{4} \qquad \text{(the original equation)}$$

Multiply each side of the equation by 5:

$$\mathbf{5}\left(\frac{3x-9}{5}\right) = \mathbf{5}\left(\frac{2x+7}{4}\right)$$

$$3x-9 = \frac{5(2x+7)}{4} \qquad \text{(since the 5's cross-cancel)}$$

This gives us

$$3x-9 = \frac{10x+35}{4}$$

Now multiply each side of the equation by 4:

$$\mathbf{4}(3x-9) = \mathbf{4}\left(\frac{10x+35}{4}\right)$$

$$\Rightarrow \quad 12x - 36 = 10x + 35 \qquad \text{(since the 4's cross-cancel)}$$

$$\Rightarrow \quad 2x - 36 = 35 \qquad \text{(subtract } 10x \text{ from each side)}$$

$$\Rightarrow \quad 2x = 71 \qquad \text{(add 36 to each side)}$$

$$\Rightarrow \quad \boxed{x = \frac{71}{2}} \qquad \text{(divide each side by 2)}$$

Homework

4. Solve each proportion (leave your answer in fractional form):

a. $\frac{x-1}{5} = \frac{2x+3}{4}$

b. $\frac{n}{3} = \frac{5-n}{10}$

c. $\frac{3a+9}{10} = \frac{2a}{3}$

d. $\frac{3}{m-2} = \frac{5}{m+12}$

e. $\frac{-2}{u-7} = \frac{-6}{9-u}$

f. $\frac{7}{-3y+7} = \frac{-1}{2-3y}$

g. $\frac{w+5}{w-3} = \frac{7}{8}$

h. $\frac{2a-1}{3} = \frac{2-a}{5}$

i. $\frac{2}{3} = \frac{4c-1}{4c+1}$

j. $\frac{-5}{2x+1} = \frac{3}{2-3x}$

k. $\frac{2y+1}{3y-4} = 8$

l. $\frac{x+3}{x-5} = \frac{x+1}{x-1}$

shortcut to solving proportions: but I will let you in on the secret only if you promise NOT mix it up with Dividing Fractions.

Solutions

1. a, b, and e are true

2. a. $x = 1.56$ b. $a = 280$ c. $b = 1.39$ d. $u = 0.21$

e. $c = 2.09$ f. $y = 37$

3. a. $a = 1.17$ b. $b = 322$ c. $n = 96$ d. $z = 1.73$

e. $u = 5.65$ f. $k = 0.2$ g. $x = -35$ h. $n = -21.25$

i. $a = \pm 9$ j. $x = 0.43$ k. $y = 2.94$ l. $w = 0.71$

m. $z = 22.14$ n. $a = 0.1$ o. $x = \pm 6$

4. a. $x = -\frac{19}{6}$ b. $n = \frac{15}{13}$ c. $a = \frac{27}{11}$

d. $m = 23$ e. $u = \frac{15}{2}$ f. $y = \frac{7}{8}$

g. $w = -61$ h. $a = \frac{11}{13}$ i. $c = \frac{5}{4}$

j. $x = \frac{13}{9}$ k. $y = \frac{3}{2}$ l. $x = -\frac{1}{3}$

"Only the curious will learn and only the resolute will overcome the obstacles to learning. The *quest quotient* has always excited me more than the intelligence quotient."

7

Edmund Wilson

CH NN – SERIES

Σ

A mathematical ***series*** is a fancy term for **sum**, or summation. A **finite** series might be the summation

$$1 + 2 + 5 + 6 + 10 \qquad \text{(which has a } \textbf{sum} \text{ of 24)}$$

An **infinite** series might be something like this:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots$$

where the three dots (called an *ellipsis*) mean to go on and on forever (infinitely many terms being added).

Believe it or not, this *infinite series* actually has a sum of $\frac{1}{2}$.

Sometimes, if there's a nice pattern, a series can be written in "**sigma**" notation. The Greek capital letter sigma, Σ, is used to represent the summation (since sigma and sum both begin with the letter ***s***). The following examples should explain how this notation should be read and calculated.

EXAMPLE 1: **Evaluate:** $\displaystyle\sum_{k=2}^{5}(2k+1)$

Solution: The expression that determines the numbers we will add together is $2k + 1$. Below the sigma sign is the starting value of the index variable k, in this case 2; above the sigma sign is the ending value of k, in this case 5. So k starts at 2 and ends at 5, but what does k do in between? We agree that it goes up by one – in other words, k will go 2, 3, 4, and then 5. Check it out, remembering that the sigma sign, Σ, means ADD:

$$\sum_{k=2}^{5}(2k+1) = \overset{(k=2)}{(2\cdot\mathbf{2}+1)}+\overset{(k=3)}{(2\cdot\mathbf{3}+1)}+\overset{(k=4)}{(2\cdot\mathbf{4}+1)}+\overset{(k=5)}{(2\cdot\mathbf{5}+1)}$$

$$= 5 + 7 + 9 + 11$$

$$= \boxed{\mathbf{32}}$$

<u>EXAMPLE 2:</u> **Evaluate:** $\displaystyle\sum_{\boldsymbol{n=0}}^{\mathbf{3}}\boldsymbol{(n^2-n)}$

<u>Solution:</u> Does it matter that Example 1 used the index variable k and this example uses the index variable n? Not at all — the variable used makes no difference in the final answer; it's just a placeholder. The values of n will be 0, 1, 2, and 3. For each value of n we evaluate the expression $n^2 - n$. Then we add up the results.

$$\sum_{n=0}^{3}(n^2-n) = \overset{(n=0)}{(\mathbf{0}^2-\mathbf{0})}+\overset{(n=1)}{(\mathbf{1}^2-\mathbf{1})}+\overset{(n=2)}{(\mathbf{2}^2-\mathbf{2})}+\overset{(n=3)}{(\mathbf{3}^2-\mathbf{3})}$$

$$= 0 + 0 + 2 + 6$$

$$= \boxed{\mathbf{8}}$$

Homework

Evaluate each series (calculate each sum):

1. $\displaystyle\sum_{k=3}^{5}(7k-1)$
2. $\displaystyle\sum_{n=2}^{5}(n^2+n)$
3. $\displaystyle\sum_{k=1}^{6}\frac{1}{2^k}$
4. $\displaystyle\sum_{j=0}^{4}3^j$
5. $\displaystyle\sum_{n=16}^{16}\frac{1}{2}\sqrt{n}$
6. $\displaystyle\sum_{k=3}^{5}\frac{1}{k}$

7. $\displaystyle\sum_{m=0}^{2} \frac{1}{m+1}$

8. $\displaystyle\sum_{t=-1}^{4} 2t$

9. $\displaystyle\sum_{j=-2}^{2} j^3$

For an extra challenge, write each summation in Σ-notation; use *k* as the index variable, and start at *k* = 1:

10. $10+11+12+13$

11. $4^3 + 5^3 + 6^3$

12. $1+\sqrt{2}+\sqrt{3}+2$

13. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

14. $1+\sqrt[3]{2}$

15. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$

Review Problems

16. $\displaystyle\sum_{n=2}^{5} (n^2+n) \; =$

17. $\displaystyle\sum_{k=1}^{6} \frac{1}{2^k} \; =$

18. $\displaystyle\sum_{j=0}^{4} 3^j \; =$

19. $\displaystyle\sum_{n=16}^{16} \tfrac{1}{2}\sqrt{n} \; =$

20. $\displaystyle\sum_{k=1}^{10} k \; =$

21. $\displaystyle\sum_{n=0}^{4} 2^{n-2} \; =$

22. True/False:

a. $\displaystyle\sum_{i=2}^{5} i^2 \; = \; 54$

b. $\displaystyle\sum_{j=0}^{3} 2^j \; = \; 16$

c. $\displaystyle\sum_{k=-1}^{1} k^3 \; = \; 3$

d. $\displaystyle\sum_{L=0}^{4} \sqrt{L} \; = \; 3+\sqrt{2}+\sqrt{3}$

23. [Only if you've studied *Combinations*]:

$$\sum_{k=0}^{4}\binom{4}{k}a^{4-k}b^{k} \;=$$

This can also be written $\sum_{k=0}^{4} C(4,k)\,a^{4-k}\,b^{k}$.

☐ *TO ∞ AND BEYOND*

A. Evaluate: $\sum_{k=1}^{99}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

B. Evaluate: $\sum_{k=1}^{\infty}\frac{1}{2^k}$

C. Evaluate: $\sum_{k=1}^{\infty}\frac{1}{k}$

[Research]

D. Evaluate: $\sum_{n=1}^{4}\frac{(-1)^{n+1}}{n^3}$

[A decimal answer will be fine.]

E. Write in Σ-notation. Use k for the index variable, starting with $k = 0$.

$$\frac{1}{2}+\frac{3}{3}+\frac{5}{6}+\frac{7}{11}+\frac{9}{18}+\frac{11}{27}$$

Solutions

1. 81

2. 68

3. $\frac{63}{64}$

4. 121

5. 2

6. $\frac{47}{60}$

7. $\frac{11}{6}$

8. 18

9. 0

10. $\sum_{k=1}^{4}(k+9)$

11. $\sum_{k=1}^{3}(k+3)^3$

12. $\sum_{k=1}^{4}\sqrt{k}$

13. $\sum_{k=1}^{5}\frac{1}{k}$

14. $\sum_{k=1}^{2}\sqrt[3]{k}$

15. $\sum_{k=1}^{4}\frac{1}{2^{k+1}}$

16. 68

17. $\frac{63}{64}$

18. 121

19. 2

20. 55

21. $\frac{31}{4}$

22. a. T b. F c. F d. T

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which can also be written $(a + b)^4$

Check out: **Math is Fun - Infinite Series**
https://www.mathsisfun.com/algebra/infinite-series.html

“The highest activity a human
being can attain is learning for
understanding, because to
understand is to be free.”

– Baruch Spinoza

1

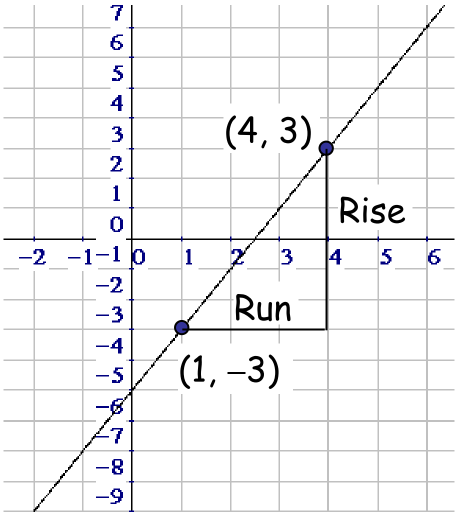
SLOPE = $\Delta y/\Delta x$

Finding the slope, $m = \frac{\text{rise}}{\text{run}}$, of a line by plotting two points and counting the squares to determine the rise and the run works fine only when it's convenient to plot the points and you're in the mood to count squares. Indeed, consider the line connecting the two points (π, 2000) and (3π, –5000). Certainly these points determine a line, and that line has some slope, but plotting these points is not feasible. We need a simpler way to calculate slope.

☐ A NEW VIEW OF SLOPE

Recall Example 1 from the chapter *Slope = Rise/Run*,

$$y = 2x - 5$$



We plotted the points (4, 3) and (1, –3) and then counted squares (as we moved from left to right) to get a rise of 6 and a run of 3, giving us a slope of

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \mathbf{2}$$

How can we get the numbers 6 and 3 <u>without</u> referring to the points on the graph? Notice that if we subtract the y-coordinate of one point from the

y-coordinate of the other point, we get

$$\text{rise} = 3 - (-3) = 3 + 3 = 6$$

Similarly, if we subtract one x-coordinate from the other, we get

$$\text{run} = 4 - 1 = 3$$

Now, dividing the rise by the run gets us our slope of 2. We can now think of our $m = \frac{\text{rise}}{\text{run}}$ formula as

$$\boxed{m = \frac{\text{change in } y}{\text{change in } x}}$$

The only issue we need to worry about is that we are consistent in the direction in which we do our subtractions. For example, using the same two points, (1, –3) and (4, 3), we can subtract in the reverse order from above, as long as <u>both</u> subtractions are reversed.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-3-3}{1-4} = \frac{-6}{-3} = \mathbf{2}$$

the same value of slope we calculated before.

<u>EXAMPLE 1</u>: **Find the slope of the line connecting the points (–7, –13) and (12, –10). Calculate the slope again by subtracting in the reverse direction.**

<u>Solution</u>: Subtracting in one direction computes the slope as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-13-(-10)}{-7-12} = \frac{-13+10}{-7-12} = \frac{-3}{-19} = \mathbf{\frac{3}{19}}$$

Reversing the direction in which we subtract the coordinates:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-10-(-13)}{12-(-7)} = \frac{-10+13}{12+7} = \mathbf{\frac{3}{19}}$$

Either way, we get the same slope; thus, the order in which you subtract is entirely up to you, as long as each subtraction (top and bottom) is done in the same direction.

New Notation

We're just about ready to find the slope of a line using the points mentioned at the beginning of this section: (π, 2000) and (3π, –5000). But first we introduce some new notation.

The natural world is filled with changes. In slope, we've seen changes in x and y in the notions of rise and run. In chemistry, there are changes in the volume and pressure of a gas. In nursing, there are changes in temperature and blood pressure, and in economics there are changes in supply and demand. This concept occurs so often that there's a special notation for a "change" in something. We use the Greek capital letter delta, **Δ**, to represent a change in something. A change in volume might be denoted by ΔV and a change in time by Δt. And so now we can redefine ***slope*** as

$$\boxed{m = \frac{\Delta y}{\Delta x}}$$

Slope is the *ratio* of the change in *y* to the change in *x*.

which is, of course, just fancy notation for what we already know.

EXAMPLE 2: **Find the slope of the line connecting the points (π, 2000) and (3π, –5000).**

Solution: A simple ratio calculation will give us the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2000-(-5000)}{\pi - 3\pi} = \frac{7000}{-2\pi} = \frac{\cancel{2}\cdot 3500}{-\cancel{2}\pi} = -\frac{\mathbf{3500}}{\pi}$$

Slope = $\Delta y / \Delta x$

In the last step of this calculation, we used the fact that a positive number divided by a negative number is negative. Also, we could obtain an approximate answer by dividing 3500 by 3.14, then attaching the negative sign, to get a slope of about **–1,114.65**.

Notice that there's no need to plot points and count squares on a grid. We've turned the geometric concept of slope into an arithmetic problem. Try reversing the order of the above subtractions to ensure you get the same slope.

Δ

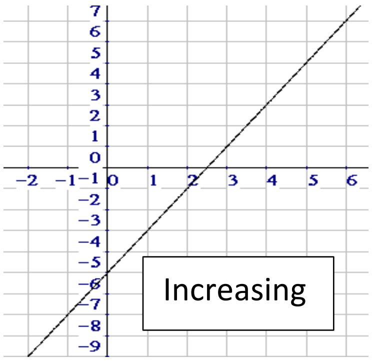
Why use delta, Δ, to represent a *change* in something? Because "delta" begins with a *d*, and *d* is the first letter of the word "difference," and difference means "subtract," and subtract is what you do when you want to calculate the change in something. (And it's the logo for Delta Airlines!)

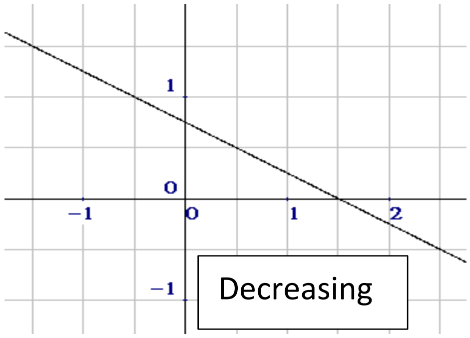
Homework

1. Use the formula $m = \dfrac{\Delta y}{\Delta x}$ to find the slope of the line connecting the given pair of points:

   a. (2, 3) and (4, 7)
   b. (–3, 0) and (0, 6)
   c. (1, –3) and (–2, 5)
   d. (2, 2) and (7, 7)
   e. (–3, –3) and (0, 0)
   f. (–1, –2) and (3, –5)
   g. (1, 1) and (–2, 3)
   h. (1, 4) and (0, 0)
   i. (–3, –2) and (1, –3)
   j. (–1, 3) and (1, –3)
   k. (–4, 5) and (0, 0)
   l. (–1, –1) and (4, –2)

❑ THE SLOPES OF INCREASING AND DECREASING LINES





Looking back at Example 1 from the chapter *Slope – Rise/Run*, let's make a quick sketch of the line. We can call this an "increasing" line because *as we move from left to right*, the line is rising, or going up, since the y-values are getting bigger. Now notice that the slope of this line, as calculated before, was 2, a positive number.

Referring to Example 2 from that chapter, we find that its graph, unlike the previous one, is falling *as we move from left to right* — that is, we have a "decreasing" line. And this is because the y-values are getting smaller. Next, we note that the slope was calculated to be the negative number $-\frac{1}{2}$.

This connection between the "increasing/decreasing" of a line and the sign of its slope is always true. Our conclusion is:

An increasing line has a positive slope, while a decreasing line has a negative slope.

Homework

2. First find the slope of the line connecting the given pair of points. Then use that slope to determine whether the graph of the line is *increasing* or *decreasing*.

   a. (–10, 7) and (–12, –8)
   b. (12, –10) and (8, –5)
   c. (12, 3) and (–3, 10)
   d. (1, 3) and (10, 5)
   e. (–8, 10) and (12, 8)
   f. (–9, 1) and (–10, 11)
   g. (–2, –1) and (1, 5)
   h. (6, –1) and (–12, –1)
   i. (4, 6) and (9, –5)
   j. (3, –3) and (12, 6)

Solutions

1. a. 1 b. 2 c. –2 d. 3 e. –3 f. –1

g. $-\frac{1}{2}$ h. $\frac{2}{3}$ i. 3 j. $\frac{3}{2}$ k. $-\frac{2}{5}$ l. $\frac{3}{4}$

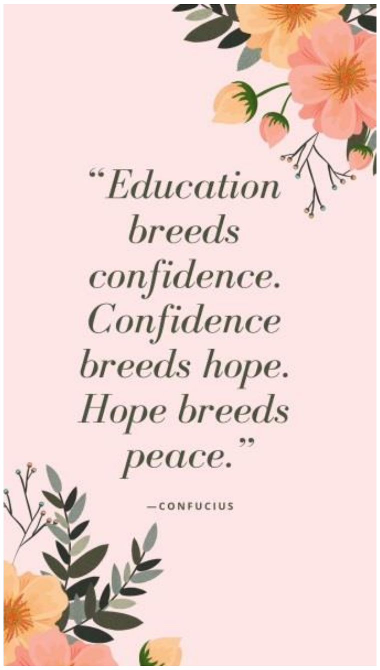
2. If the slope is positive, the line is increasing; if the slope is negative, the line is decreasing. But what about part h. of this problem?

a. $\frac{15}{2}$; inc b. $-\frac{5}{4}$; dec c. $-\frac{7}{15}$; dec

d. $\frac{2}{9}$; inc e. $-\frac{1}{10}$; dec f. –10; dec

g. 2; inc h. 0; ??? i. $-\frac{11}{5}$; dec

j. 1; inc



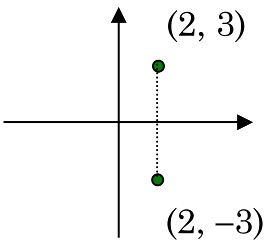
"Education breeds confidence. Confidence breeds hope. Hope breeds peace."

—CONFUCIUS

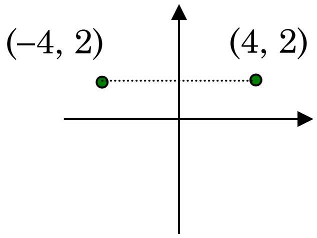
CH N – SYMMETRY, EVEN AND ODD FUNCTIONS

INTRODUCTION

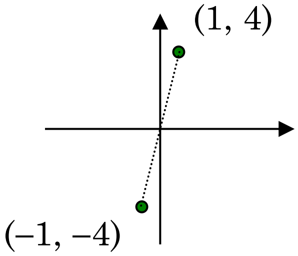
SYMMETRY OF A PAIR OF POINTS



x-axis symmetry



y-axis symmetry



origin symmetry

In the first graph, notice that (2, 3) is directly above (2, –3), and that the two points are equidistant (equally distant) from the x-axis; that is, <u>the dotted line is bisected by the x-axis</u>. It's as though the bottom point is a reflection in the water of the top point. Also, note that the x-coordinates of the two points are the same and their y-coordinates are opposites. The two points are ***symmetric with respect to the x-axis***.

In the second graph, we see that (4, 2) is directly to the right of (–4, 2), and that the two points are equidistant from the y-axis; that is, <u>the dotted line is bisected by the y-axis</u>. It's like each point is a mirror image of the other. The two points are ***symmetric with respect to the***

y-axis. Also, can you see that the x-coordinates of the two points are opposites, and their y-coordinates are the same?

In the third graph, note that the two points are equidistant from the origin, and the dotted line is bisected by the origin. The two points are ***symmetric with respect to the origin***. Lastly, it should be clear that the x-coordinates of the two points are opposites, and their y-coordinates are also opposites.

Homework

1. For each of the following points, find the point which has x-axis symmetry with it:

   a. $(2, 7)$ b. $(-3, 4)$ c. $(5, \pi)$ d. $(4, 0)$ e. $(0, -3)$

2. For each of the following points, find the point which has y-axis symmetry with it:

   a. $(-2, 5)$ b. $(-1, -2)$ c. $(7, 0)$ d. $(\pi, -\sqrt{2})$ e. $(0, -2)$

3. For each of the following points, find the point which has origin symmetry with it:

   a. $(-3, 5)$ b. $(8, -7)$ c. $(0, 0)$ d. $(6, 0)$ e. $(0, -\pi)$

4. The line segment connecting the points P and Q is bisected by the y-axis. Does this imply that P and Q must be symmetric?

5. Let P be a point in the first quadrant and let Q be a point such that P and Q have x-axis symmetry. Then Q must lie in which quadrant?

6. Let (x, y) be a point in the third quadrant. Name the point such that the pair of points has origin symmetry.

Solutions

1. a. $(2, -7)$ b. $(-3, -4)$ c. $(5, -\pi)$ d. $(4, 0)$ e. $(0, 3)$

2. a. $(2, 5)$ b. $(1, -2)$ c. $(-7, 0)$ d. $(-\pi, -\sqrt{2})$ e. $(0, -2)$

3. a. $(3, -5)$ b. $(-8, 7)$ c. $(0, 0)$ d. $(-6, 0)$ e. $(0, \pi)$

4. No **5**. IV **6**. $(-x, -y)$

CH NN – 3×3 SYSTEMS OF LINEAR EQUATIONS, MORE PARABOLAS

There are many problems in math, science, and business that require more than two variables. Here we will solve a system of three equations with three variables. Note that a solution will consist of a set of three numbers that will satisfy all three of the equations. We will use these ideas to find the path of a missile.

❑ *EXAMPLE*

Solve the system: $\mathbf{4x - 2y + 3z = -14}$ [Equ 1]

$\mathbf{6x + y - z = 13}$ [Equ 2]

$\mathbf{-x + 3y - 4z = 24}$ [Equ 3]

Start with Equ 1 and Equ 2 to eliminate the x:

$4x - 2y + 3z = -14$ (times 3) $\Rightarrow$ $12x - 6y + 9z = -42$

$6x + y - z = 13$ (times –2) $\Rightarrow$ $-12x - 2y + 2z = -26$

Add: $-8y + 11z = -68$ [Equ 4]

Now use Equ 1 and Equ 3 to eliminate the x:

$4x - 2y + 3z = -14$ (leave it) $\Rightarrow$ $4x - 2y + 3z = -14$

$-x + 3y - 4z = 24$ (times 4) $\Rightarrow$ $-4x + 12y - 16z = 96$

Add: $10y - 13z = 82$ [Equ 5]

Solve the 2×2 system consisting of Equ 4 and Equ 5:

$$\begin{array}{llll}
\text{[Equ 4]} & -8y + 11z = -68 & \text{(times 5)} & \Rightarrow -40y + 55z = -340 \\
\text{[Equ 5]} & 10y - 13z = 82 & \text{(times 4)} & \Rightarrow \quad 40y - 52z = 328 \\
\hline
 & & \text{Add:} & \quad 3z = -12 \\
 & & & \Rightarrow \boxed{z = -4}
\end{array}$$

We now calculate y using Equ 4 and our newly found value of z:

$$-8y + 11z = -68$$

$$\Rightarrow \quad -8y + 11(\underline{\mathbf{-4}}) = -68$$

$$\Rightarrow \quad -8y - 44 = -68$$

$$\Rightarrow \quad -8y = -24$$

$$\Rightarrow \quad \boxed{y = 3}$$

And lastly, we solve for x using Equ 1 and our values of y and z:

$$4x - 2y + 3z = -14$$

$$\Rightarrow \quad 4x - 2(\underline{\mathbf{3}}) + 3(\underline{\mathbf{-4}}) = -14$$

$$\Rightarrow \quad 4x - 6 - 12 = -14$$

$$\Rightarrow \quad 4x - 18 = -14$$

$$\Rightarrow \quad 4x = 4$$

$$\Rightarrow \quad \boxed{x = 1}$$

We conclude that the solution of the system of 3 equations in 3 variables is

$$\boxed{x = 1,\ y = 3,\ z = -4}$$

We could CHECK our solution by plugging the values of x, y, and z into <u>ALL THREE</u> of the *original* equations.

Homework

Solve each system of equations:

1. $$\begin{aligned} 2x - y + 3z &= 9 \\ x + 4y - z &= 6 \\ -3x + 2y + 2z &= 7 \end{aligned}$$

2. $$\begin{aligned} 3a - 4b + c &= 9 \\ -2a - 4b + 3c &= 19 \\ 5a + 3b - c &= -8 \end{aligned}$$

3. $$\begin{aligned} 2p + 3q - r &= 2 \\ -p - 3r + 2r &= 5 \\ 7p - 4p - 3r &= 23 \end{aligned}$$

4. $$\begin{aligned} 7x - 2y + 3z &= -12 \\ -5x + 7y - 4z &= 3 \\ 6x + 3y - 2z &= 6 \end{aligned}$$

❑ FINDING THE PARABOLA THROUGH THREE POINTS

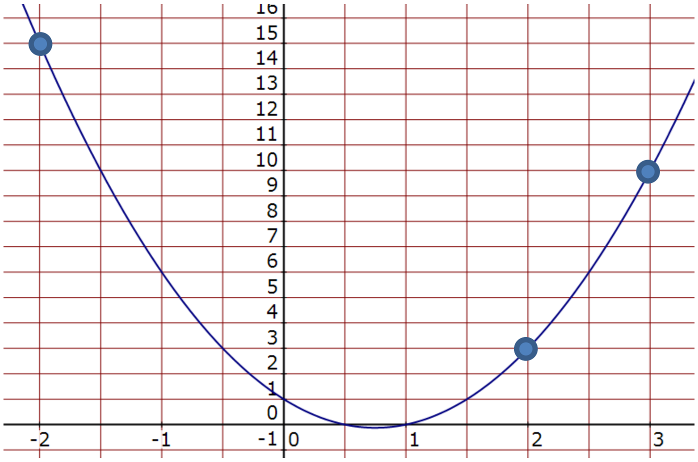
EXAMPLE 3: **Find the equation of the parabola which passes through the three points (2, 3), (3, 10), and (–2, 15).**

Solution: Recall that the parabolas in this class have the form

$$y = ax^2 + bx + c$$

To fully reveal the identity of the unknown parabola, we need to find the values of a, b, and c.

Now, what does it mean for a certain point to lie on the graph of some equation? It means that the coordinates of that point (the x and y)



should work in the equation.

Therefore, if the point (2, 3) is to lie on the parabola $y = ax^2 + bx + c$, we should be able to plug 2 in for x and plug 3 in for y — yielding the equation:

$$\mathbf{3} = a \cdot \mathbf{2}^2 + b \cdot \mathbf{2} + \text{c}$$

or, $\underline{3 = 4a + 2b + c}$

Since (3, 10) is on the parabola, its coordinates should also satisfy the parabola equation:

$$\mathbf{10} = a \cdot \mathbf{3}^2 + b \cdot \mathbf{3} + c$$

or, $\underline{10 = 9a + 3b + c}$

And last, using the point (–2, 15) yields the equation:

$$\mathbf{15} = a \cdot (-\mathbf{2})^2 + b \cdot -\mathbf{2} + c$$

or, $\underline{15 = 4a - 2b + \text{c}}$

Let's summarize what we have by taking all three underlined equations and writing them together as a system of three equations in the three variables a, b, and c. We'll also flip each equation around.

$$\begin{array}{ll} 4a + 2b + c = 3 & \text{[Equ 1]} \\ 9a + 3b + c = 10 & \text{[Equ 2]} \\ 4a - 2b + c = 15 & \text{[Equ 3]} \end{array}$$

We now have a 3×3 system of equations, which we can solve as we did in the previous section.

Start with Equ 1 and Equ 2 and eliminate the a:

$$\begin{array}{lcr} a + b + c = 0 \;\; \textbf{(times –9)} & \Rightarrow & -9a - 9b - 9c = 0 \\ 9a + 3b + c = 10 & \Rightarrow & \underline{9a + 3b + \text{c} = 10} \\ & & -6b - 8c = 10 \quad \text{[Equ 4]} \end{array}$$

Now use Equ 1 and Equ 3 and eliminate the a:

$$\begin{array}{rcl} a+b+c = 0 \ \textbf{(times –4)} & \Rightarrow & -4a-4b-4c = 0 \\ 4a-2b+c = 15 & \Rightarrow & \underline{4a-2b+c = 15} \\ & & -6b-3c = 15 \quad \text{[Equ 5]} \end{array}$$

Solve the 2×2 system consisting of Equ 4 and Equ 5:

$$\begin{array}{lrcl} \text{[Equ 4]} & -6b-8c = 10 \ \textbf{(times –1)} & \Rightarrow & 6b+8c = -10 \\ \text{[Equ 5]} & -6b-3c = 15 & \Rightarrow & \underline{-6b-3c = 15} \\ & & & 5c = 5 \\ & & & \boxed{c = 1} \end{array}$$

Using Equ 4:

$$\begin{array}{rl} & -6b-8c = 10 \\ \Rightarrow & -6b-8(\mathbf{1}) = 10 \\ \Rightarrow & -6b-8 = 10 \\ \Rightarrow & -6b = 18 \\ \Rightarrow & \boxed{b = -3} \end{array}$$

(Equ 5 would have worked just as well)

Using Equ 1 from way back, we calculate the value of a:

$$\begin{array}{rl} & a+b+c = 0 \\ \Rightarrow & a-\mathbf{3}+\mathbf{1} = 0 \\ \Rightarrow & a-2 = 0 \\ \Rightarrow & \boxed{a = 2} \end{array}$$

(Equ 2 or 3 would have worked just as well)

Remember, we started with the parabola $y = ax^2 + bx + c$, plugged in the three given points, and determined that $a = 2$, $b = -3$, and $c = 1$. We're done — the parabola we're seeking is

$$\boxed{y = 2x^2 - 3x + 1}$$

Homework

5. Find the equation of the **parabola** passing through the three given points:

a. (1, 4) (2, 6) (3, 10)

b. (–1, –2) (2, 13) (–3, 8)

c. (3, 13) (1, 3) (–2, 3)

d. (1, 3) (2, –7) (–1, 5)

e. (1, –1) (–1, –7) (2, –10)

f. (1, 17) (2, 38) (–3, 73)

g. (1, 1) (2, –5) (–2, 7)

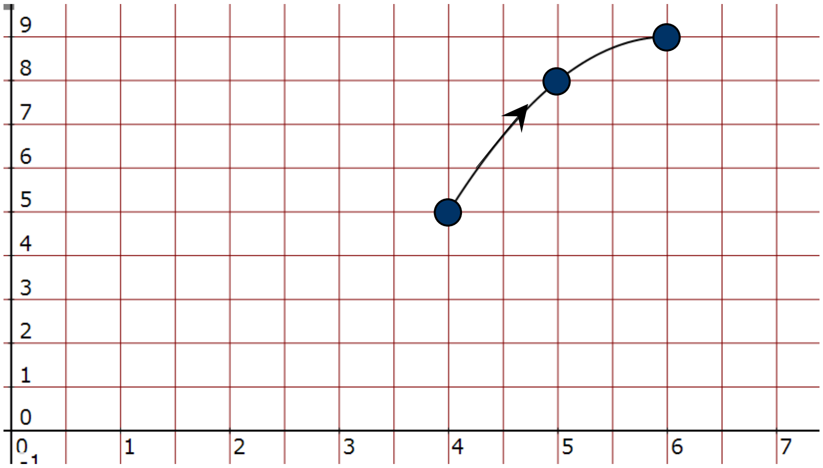
h. (1, 3) (–1, 13) (2, 22)

i. (1, –5) (–2, –11) (–1, –5)

j. (1, 7) (–2, 52) (–1, 21)

<u>EXAMPLE 4:</u> **A missile is fired from an unknown location on the x-axis of the radar grid. Its first three positions are reported as (4, 5), (5, 8), and (6, 9), at which time the radar jams. Assume that the missile follows a parabolic trajectory (path), and that the missile is traveling east. Calculate both the x-value (on the x-axis) from where the missile was fired and the x-value (on the x-axis) where the missile will hit the ground.**

<u>Solution:</u> Here's a graphic of the missile's trajectory, as determined by the three reported positions:



Where was it launched, and where will it land?

Our goal is to "extrapolate" (extend) this curve to a full parabola formula, so we can deduce where the missile came from (pretty important) and where it's going to land (REALLY important).

Beginning with $y = ax^2 + bx + c$ (see Example 3), we get the following three equations, obtained by plugging each of the points on the parabola into the generic parabola equation:

$$y = ax^2 + bx + c$$

$$\begin{array}{lll}
(4, 5): & \mathbf{5} = a(\mathbf{4}^2) + b(\mathbf{4}) + c & \Rightarrow \quad 16a + 4b + c = 5 \\
(5, 8): & \mathbf{8} = a(\mathbf{5}^2) + b(\mathbf{5}) + c & \Rightarrow \quad 25a + 5b + c = 8 \\
(6, 9): & \mathbf{9} = a(\mathbf{6}^2) + b(\mathbf{6}) + c & \Rightarrow \quad 36a + 6b + c = 9
\end{array}$$

We now have a system of three equations in three unknowns, which can be solved using the techniques of this chapter. When you (yes, you!) solve the system you will determine that $a = -1$, $b = 12$, and $c = -27$, from which we deduce that the parabola which describes the trajectory is

$$y = -x^2 + 12x - 27$$

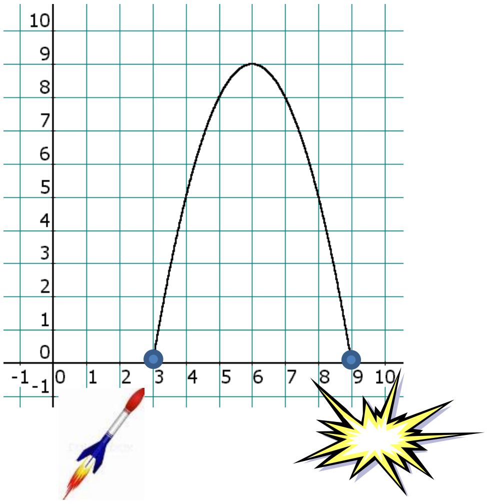
But we're still not done. (By the way, we will not be blown to bits by the missile while we wade through all these calculations. A computer can solve the problem in a billionth of a second – but remember: someone has to program the computer.) To find the firing point and the landing point on the x-axis, we now calculate the two x-intercepts of the parabola, which we do by setting y to 0:

$$\begin{array}{ll}
 & 0 = -x^2 + 12x - 27 \\
\Rightarrow & x^2 - 12x + 27 = 0 \\
\Rightarrow & (x - 3)(x - 9) = 0 \\
\Rightarrow & x - 3 = 0 \text{ or } x - 9 = 0 \\
\Rightarrow & x = 3 \text{ or } x = 9 \Rightarrow
\end{array}$$

The x-intercepts are **(3, 0)** and **(9, 0)**.

We're now ready to answer the question at hand:

The missile was fired from $\boldsymbol{x} = \mathbf{3}$, and will land at $\boldsymbol{x} = \mathbf{9}$.



Homework

6. Solve the missile trajectory problem if the missile was spotted at the points (4, 14), (5, 18), (7, 20) before the radar failed.

7. Solve the missile trajectory problem if the missile was spotted at the points (7, 10), (8, 12) and (9, 12).

8. Solve the missile trajectory problem if the missile was spotted at the points (1, 0), (3, 8) and (5, 8).

9. Solve the missile trajectory problem if the missile was spotted at the points (3, 3), (4, 4) and (5, 3).

Solutions

1. $x = 1, y = 2, z = 3$

2. $a = 0, b = -1, c = 5$

3. $p = 4, q = -1, r = 3$

4. $x = -1, y = -2, z = -3$

5. a. $y = x^2 - x + 4$ b. $y = 2x^2 + 3x - 1$ c. $y = x^2 + x + 1$
d. $y = -3x^2 - x + 7$ e. $y = -4x^2 + 3x$ f. $y = 7x^2 + 10$
g. $y = -x^2 - 3x + 5$ h. $y = 8x^2 - 5x$ i. $y \; -2x^2 - 3$
j. $y = 8x^2 - 7x + 6$

6. Fired from $x = 2$; will land at $x = 11$

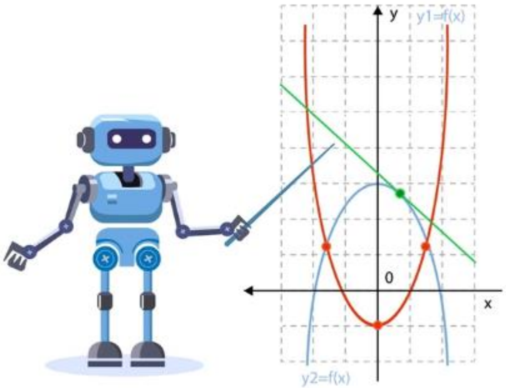
7. Fired from $x = 5$; will land at $x = 12$

8. Fired from $x = 1$; will land at $x = 7$

9. Fired from $x = 2$; will land at $x = 6$

CH NN – SYSTEMS OF LINEAR EQUATIONS, ELIMINATION

To solve the applications in future chapters, we need to be able to solve a system of two equations in two variables. A method that works very well in many cases is called the ***Elimination Method***. We multiply one or both equations by appropriate numbers (whatever that means), **add** the resulting equations to eliminate a variable, and then solve for the variable that survived.



❑ EXAMPLES

EXAMPLE 1: **Solve the system:**

$$\begin{aligned} 5x - 2y &= 20 \\ 3x + 7y &= -29 \end{aligned}$$

Solution: In the Elimination Method we may eliminate either variable. But there's a certain "orderliness" that comes in handy in future math courses if we always eliminate the first variable, so in this case we will eliminate the x. As mentioned above, we multiply one or both equations by some numbers, add the resulting equations to kill off one of the variables, and then solve for the variable that still lives. How do we find these numbers? Rather than some mystifying explanation, just watch — you'll catch on.

$$\begin{aligned} 5x-2y &= 20 \xrightarrow{\text{times 3}} & 15x-6y &= 60 \\ 3x+7y &= -29 \xrightarrow{\text{times -5}} & -15x-35y &= 145 \end{aligned}$$

Add the equations: $0x - 41y = 205$

The x's are gone!

Divide by –41: $\frac{-41y}{-41} = \frac{205}{-41} \Rightarrow \underline{y = -5}$

Now that we have the value of y, we can substitute its value of –5 into either of the two original equations to find the value of x. Using the first equation:

$$\begin{aligned} & 5x - 2(\mathbf{-5}) = 20 \\ \Rightarrow\ & 5x + 10 = 20 \\ \Rightarrow\ & 5x = 10 \\ \Rightarrow\ & \underline{x = 2} \end{aligned}$$

Therefore, the final solution to the system of equations is

$$\boxed{x = 2 \ \& \ y = -5}$$

<u>EXAMPLE 2:</u> **Solve the system:** $\mathbf{12a + 7b = 9}$
$\mathbf{-18a - 5b = 3}$

<u>Solution:</u> Notice how the numbers we choose to multiply each equation by will accomplish our goal: They make the a's disappear when the equations are added.

$$\begin{aligned} 12a+7b &= 9 \xrightarrow{\text{times 3}} & 36a+21b &= 27 \\ -18a-5b &= 3 \xrightarrow{\text{times 2}} & -36a-10b &= 6 \end{aligned}$$

Add the equations: $0a + 11b = 33$

Divide by 11: $\underline{b = 3}$

Placing this value of b into the first equation gives us

$$12a + 7(\mathbf{3}) \ = \ 9 \quad \Rightarrow \quad 12a + 21 \ = \ 9 \quad \Rightarrow \quad 12a \ = \ -12 \quad \Rightarrow \quad \underline{a = -1}$$

We now have our complete solution to the system of equations:

$$\boxed{a \ = \ -1 \ \ \& \ \ b \ = \ 3}$$

Let's use this example to learn how to check our solution. The main theme is this: The values of a and b must work in both of the original equations in order to constitute a valid solution.

1st equation:

$$12a + 7b \ = \ 9$$

$$12(\mathbf{-1}) + 7(\mathbf{3}) \ \overset{?}{=} \ 9$$

$$-12 + 21 \ \overset{?}{=} \ 9$$

$$9 \ = \ 9 \ ✓$$

2nd equation:

$$-18a - 5b \ = 3$$

$$-18(\mathbf{-1}) - 5(\mathbf{3}) \ \overset{?}{=} \ 3$$

$$18 - 15 \ \overset{?}{=} \ 3$$

$$3 \ = \ 3 \ ✓$$

Our conclusion is that the values $a = -1$ and $b = 3$ work perfectly. The solution can also be written as the ordered pair (–1, 3).

Homework

1. Solve each system using the Elimination Method, and be sure you practice checking your solution (your pair of numbers) in both of the original equations:

a. $$\begin{aligned} 2x + y \ &= \ 5 \\ -2x + 7y \ &= \ 19 \end{aligned}$$

b. $$\begin{aligned} 5a - 3b \ &= \ 5 \\ 10a + 4b \ &= \ -40 \end{aligned}$$

c. $$\begin{aligned} -2u - 3v \ &= \ -16 \\ -7u + 8v \ &= \ -56 \end{aligned}$$

d. $$\begin{aligned} 7x + 12y \ &= \ -24 \\ 6x - 7y \ &= \ 14 \end{aligned}$$

e. $\begin{aligned} 3m-2n &= 34 \\ -6m+n &= -62 \end{aligned}$

f. $\begin{aligned} -3s-3t &= -24 \\ 10s+8t &= 64 \end{aligned}$

g. $\begin{aligned} 2c-3d &= 13 \\ 5c+6d &= -8 \end{aligned}$

h. $\begin{aligned} -5w-4x &= -20 \\ 20w+3x &= 15 \end{aligned}$

i. $\begin{aligned} -5x-4n &= -8 \\ 11x+6n &= -2 \end{aligned}$

j. $\begin{aligned} 2w-4a &= 6 \\ -3w+9a &= -12 \end{aligned}$

k. $\begin{aligned} 2n-3y &= -2 \\ 8n-11y &= -2 \end{aligned}$

l. $\begin{aligned} 4c+9y &= 4 \\ -5c-11y &= 12 \end{aligned}$

m. $\begin{aligned} 5g-2h &= -6 \\ 4g+2h &= 3 \end{aligned}$

n. $\begin{aligned} -4w+3h &= -1 \\ -3w+4h &= 5 \end{aligned}$

o. $\begin{aligned} -3w+4m &= 6 \\ -3w-m &= 1 \end{aligned}$

p. $\begin{aligned} 3a+3q &= 1 \\ -5a+5q &= 6 \end{aligned}$

Solutions

1. a. $x = 1, y = 3$

Complete Check:

$2x + y = 5$

$2(\mathbf{1}) + \mathbf{3} = 5$

$2 + 3 = 5$

$5 = 5$ ✓

$-2x + 7y = 19$

$-2(\mathbf{1}) + 7(\mathbf{3}) = 19$

$-2 + 21 = 19$

$19 = 19$ ✓

b. $a = -2, b = -5$

c. $u = 8, v = 0$

d. $x = 0, y = -2$

e. $m = 10, n = -2$

f. $s = 0, t = 8$

g. $c = 2, d = -3$

h. $w = 0, x = 5$

i. $x = -4, n = 7$

j. $w = 1, a = -1$

k. $n = 8, y = 6$

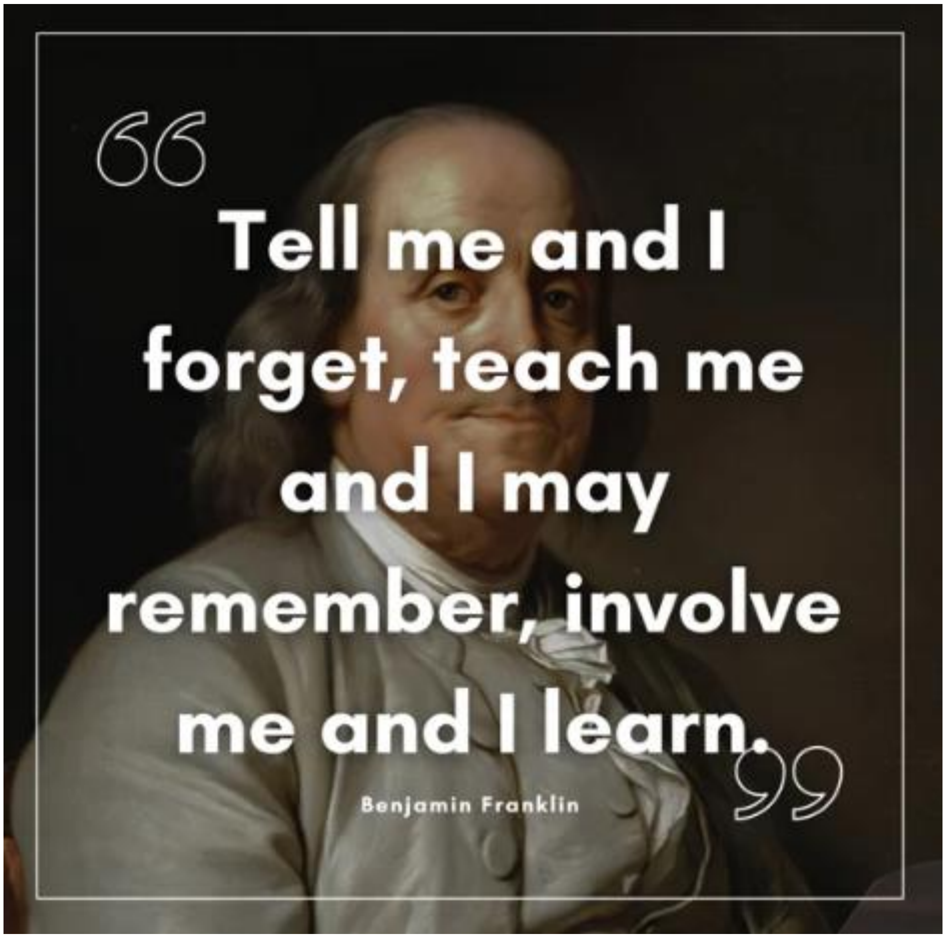
l. $c = -152, y = 68$

m. $g = -\frac{1}{3}, h = \frac{13}{6}$

n. $w = \frac{19}{7}, h = \frac{23}{7}$

o. $w = -\frac{2}{3}, m = 1$

p. $a = -\frac{13}{30}, q = \frac{23}{30}$


Tell me and I
forget, teach me
and I may
remember, involve
me and I learn.
Benjamin Franklin

CH NN – SYSTEMS OF LINEAR EQUATIONS, GRAPHING

Consider the system of two equations in two variables:

$$\begin{aligned} x+y &= 7 \\ -2x+y &= 1 \end{aligned}$$

Assuming you know how to graph straight lines, we're going to learn a method to solve a system of two linear equations in two variables. Each equation in the system above is a line; assuming the lines intersect (and assuming they're not the same line), there will be <u>one</u> point of intersection. Since that point of intersection lies on both lines (is that clear?), the coordinates (the *x* and *y*) of that point must satisfy both of the equations. Does that make any sense?

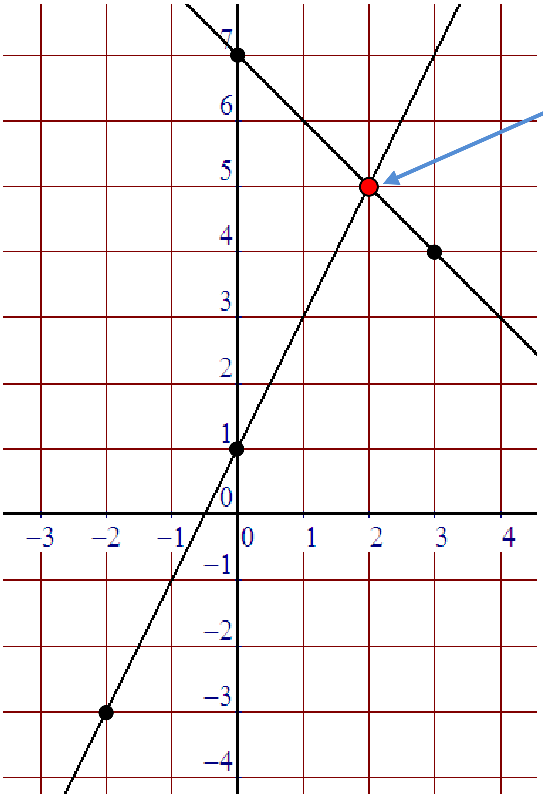
☐ *TWO LINES*

<u>EXAMPLE 1:</u> **Solve the system of equations** $\begin{aligned} \boldsymbol{x+y} &= \mathbf{7} \\ \boldsymbol{-2x+y} &= \mathbf{1} \end{aligned}$ **by graphing.**

<u>Solution:</u> Each equation is a line, so let's graph each of them and then turn our attention to their point of intersection.

Line #1: Solve for y to get $y = -x + 7$. If we let $x = 3$, then $y = 4$, and so **(3, 4)** is on the line. If we choose $x = 0$, then $y = 7$, in which case **(0, 7)** is on the line. We'll use these two points for Line #1.

Line #2: Solve for y to get $y = 2x + 1$. If $x = 0$, then $y = 1$, and if $x = -2$, then $y = -3$. So we'll use the points **(0, 1)** and **(–2, –3)** for Line #2.



The point of intersection of the two lines appears to be the point (2, 5). This means that the solution of the given system of equations is

$x = 2$, $y = 5$

Important Note: Since we're reading points on a graph, it's very easy to misread them (imagine if the x- and y-coordinates were fractions or square roots). Thus, the graphing method yields only an *approximation*, but it's a great method when algebraic methods – for instance, Elimination and Substitution – fail to work.

Homework

1.

2.

CH XX – SYSTEMS OF LINEAR EQUATIONS, SUBSTITUTION

One of the methods that's really useful in solving systems of equations is the Substitution Method, where a variable is solved for in one equation, and then its value is substituted into another equation.

We'll start with the simplest substitution problem, which turns out to be very useful in calculus.

EXAMPLE 1: **Solve the system for *x*:**

$$\begin{aligned} y &= 3x - 7 \\ y &= -9x + 8 \end{aligned}$$

Solution: Since the two expression are both equal to y, they must be equal to each other:

$$3x - 7 = -9x + 8$$

$$\Rightarrow \quad 12x = 15$$

$$\Rightarrow \quad x = \frac{15}{12}$$

$$\Rightarrow \quad \boxed{x = \frac{5}{3}}$$

By the way, we've just calculated the *x*-coordinate of the *point of interection* of the lines y = 3x –7 and y = –9x + 8.

☐ FRACTIONS NOT NEEDED

EXAMPLE 2: **Solve the system:**

$$\begin{aligned} 3x + y &= 12 \\ -5x + 2y &= 13 \end{aligned}$$

Solution: The command "Solve the system" means we must find the values of both x and y that make both equations true.

The first step is to choose an equation and a variable within that equation to solve for. For this system, the best variable to solve for is the y in the first equation. Why y? Because its coefficient is 1, which means we avoid fractions when solving for y.

Solving the first equation for y gives us

$y = 12 - 3x$ (by subtracting $3x$ from each side)

Now we substitute the expression $12 - 3x$ into the second equation where the y is.

$-5x + 2y = 13$ (the second equation)

$\Rightarrow -5x + 2(\mathbf{12 - 3x}) = 13$ (since $y = 12 - 3x$)

$\Rightarrow -5x + 24 - 6x = 13$ (distribute)

$\Rightarrow -11x + 24 = 13$ (combine the x's)

$\Rightarrow -11x = -11$ (subtract 24 from each side)

$\Rightarrow \underline{x = 1}$ (divide each side by -11)

Since $y = 12 - 3x$, and we just found out that $x = 1$, we can solve for y:

$$\begin{aligned} y &= 12 - 3(\mathbf{1}) \\ y &= 12 - 3 \\ \underline{y} &\underline{= 9} \end{aligned}$$

$$\boxed{x = 1 \ \& \ y = 9}$$

Notes: First, if you'd like to check our solution, substitute the values of x and y into both of the original equations. Second, what if none of the variables has a coefficient of 1? Well, you'll just have to pick a variable and then deal with whatever fractions may arise. In the real world, however, it's probably best to solve the system using the Elimination Method.

Homework

1. Solve each system using the Substitution Method, and be sure you practice checking your solution (your pair of numbers) in <u>both</u> of the <u>original</u> equations:

a. $\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$ b. $\begin{aligned} 3m - 2n &= 34 \\ -6m + n &= -62 \end{aligned}$ c. $\begin{aligned} -3w + 4m &= 6 \\ -3w - m &= 1 \end{aligned}$

❑ FRACTIONS NEEDED

<u>EXAMPLE 2:</u> **Solve the system:** $\begin{aligned} \mathbf{5x + 3y} &= \mathbf{7} \\ \mathbf{4x - 2y} &= \mathbf{1} \end{aligned}$

<u>Solution:</u> There's no way to avoid fractions (can you see why?), so even though we could solve for either variable in either equation, for no particular reason we'll solve for x in the first equation:

$5x + 3y = 7$ (the first equation)

$\Rightarrow 5x = -3y + 7$ (subtract $3y$ from each side)

$\Rightarrow x = \dfrac{-3y+7}{5}$ (*) (divided each side by 5)

Now substitute this value of x into the second equation:

$$4\left(\frac{-3y+7}{5}\right) - 2y = 1$$

$$\Rightarrow \quad \frac{-12y+38}{5} - 2y \;=\; 1$$

Multiply both sides of the equation (that is, all three terms) by 5:

$$\Rightarrow \quad \left(\frac{-12y+28}{5}\right)\left[5\right] - 2y\left[5\right] \;=\; 1\left[5\right]$$

$$\Rightarrow \quad \left(\frac{-12y+28}{\cancel{5}}\right)\left[\cancel{5}\right] - 2y\left[5\right] \;=\; 1\left[5\right]$$

$$\Rightarrow \quad -12y + 28 - 10y \;=\; 5$$

$$\Rightarrow \quad -22y \;=\; -23$$

$$\Rightarrow \quad y = \frac{23}{22}$$

Sorry, but we're not done — we still need to find x, so we substitute the value of y just obtained into the equation for x (the one with the * next to it):

$$x \;=\; \frac{-3y+7}{5}$$

And since $y = \frac{17}{22}$, it follows that (hold on tight!)

$$x \;=\; \frac{-3\left(\frac{23}{22}\right)+7}{5} \;=\; \frac{\frac{-69}{22}+7}{5} \;=\; \frac{\frac{-69}{22}+\frac{7(\mathbf{22})}{\mathbf{22}}}{5} \;=\; \frac{\frac{-69}{22}+\frac{154}{22}}{5} \;=\; \frac{\frac{85}{22}}{5}$$

$$= \;\frac{85}{22}\times\frac{1}{5} \;=\; \frac{5\cdot 17}{22}\times\frac{1}{5} \;=\; \frac{\cancel{5}\cdot 17}{22}\times\frac{1}{\cancel{5}} \;=\; \frac{17}{22}$$

And so our final solution is

$$\boxed{x = \frac{17}{22} \text{ and } y = \frac{23}{22}}$$

Homework

2. Solve each system using the Substitution Method:

a. $\begin{aligned} 2x + 3y &= 7 \\ 4x - 2y &= 11 \end{aligned}$

b. $\begin{aligned} 5a - 2b &= 5 \\ 4a - 3b &= 9 \end{aligned}$

c. $\begin{aligned} 2w + 3z &= 7 \\ w - 3z &= 13 \end{aligned}$

Solutions

1. a. $x = 1, y = 3$ b. $m = 10, n = -2$ c. $w = -\frac{2}{3}, m = 1$

2. a. $x = \frac{47}{16}, \ y = \frac{3}{8}$ b. $a = -\frac{3}{7}, \ b = \frac{-25}{7}$ c. $w = \frac{20}{3}, \ z = \frac{-19}{9}$

“There are no shortcuts to any place worth going.”

– Beverly Sills, Opera Soprano

CH XX – DIRECT AND INVERSE VARIATION

☐ *INTRODUCTION*

Let's start right off with an example:

Let D = number of hard ***D***rives sold
C = the ***C***apacity of the drive, in gigabytes (GB)
P = selling ***P***rice

Now let's make up a formula that will illustrate the ideas of this chapter:

$$D = \frac{5C}{P}$$

① What happens to drives sold, *D,* when the capacity, *C,* is increased? Let's assume that the drive capacity was 200 GB last month and the price was $100. The number of drives sold was

$$D = \frac{5C}{P} = \frac{5 \cdot 200}{100} = \underline{\text{10 drives}}$$

Now increase the capacity to 600 GB; the number of drives sold will be

$$D = \frac{5C}{P} = \frac{5 \cdot \mathbf{600}}{100} = \underline{\text{30 drives}}$$

Make sense? Increasing the capacity (allowing the drives to hold more data) and keeping the price at $100 should <u>increase</u> sales.

② Now let’s see what happens if we increase the price and keep the capacity the same. Assume that last month the price, P, was \$150 when the capacity was 450 GB. We calculate the number of drives sold:

$$D = \frac{5C}{P} = \frac{5 \cdot 450}{150} = \underline{15 \text{ drives}}$$

Let’s predict future sales if we increase the price from \$150 to \$225:

$$D = \frac{5C}{P} = \frac{5 \cdot 450}{225} = \underline{10 \text{ drives}}$$

Make sense? Increasing the price and keeping the capacity the same produced a <u>decrease</u> in sales.

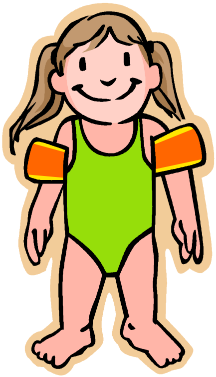
Without going into the details, we could also demonstrate that if the capacity, C, goes down, so will D, the number of drives sold. And if the price, P, goes down, D will go up.

❑ DIRECT AND INVERSE VARIATION

<u>Direct Variation</u> What happens to the sale of bathing suits, B, when the temperature, T, goes up? As the temperature rises, so does the sale of bathing suits. We say that B is ***directly proportional*** to T, or that B ***varies directly*** as T. A formula of this type might be

$$B = 3T$$

For example, if the temperature is 80°, then 240 suits will be sold. But if the temperature rises to 100°, then 300 units will be sold. When it gets cold again, B will decrease. Whatever T does (increase or decrease), B does the **same**.



<u>Inverse Variation</u> What happens to the air pressure, P, as you increase your elevation, E? As your elevation goes up, the air pressure goes down. We say that P is

inversely proportional to E, or that P ***varies inversely*** as E. An example might be the formula

$$P = \frac{2000}{E}$$

For instance, if $E = 200$, then $P = 10$. But if E is increased to 500, then the pressure P is reduced to 4. When you return to lower altitudes, the pressure goes back up. Whatever E does, P does the **reverse**.

See the numbers "3" and "2000" in our two formulas above? Each of these numbers is called a ***constant of proportionality*** (or ***constant of variation***).

In the following definitions, the letter ***k*** is the <u>positive</u> constant of proportionality (or constant of variation).

$y = kx$ is read "y is ***directly proportional to*** x," or "y ***varies directly as*** x," and means: If x increases, then y increases; and if x decreases, then y decreases. In other words, y does whatever x does.

$y = \frac{k}{x}$ is read "y is ***inversely proportional to*** x," or "y ***varies inversely as*** x," and means: If x increases, then y decreases; and if x decreases, then y increases. In other words, y does the <u>reverse</u> of what x does.

Homework

1. Consider the *direct variation* $D = 12Q$. We can pretend that Q is the quality of a car and D is the demand for that car.

   a. Find the value of D if Q is 40.

b. Double the Q to 80 and recalculate D.
c. When the quality increased, what happened to the demand?
d. Now reduce Q to 3 and recalculate D.
e. As the quality decreased, what happened to the demand?

2. Consider the *inverse variation* $P = \dfrac{200}{S}$. Let's assume that S is supply and P is price.

a. Find the value of P if $S = 10$.
b. Quadruple the S to 40 and recalculate P.
c. When the supply increased, what happened to the price?
d. Now reduce S to 5 and recalculate P.
e. As the supply decreased, what happened to the price?

☐ *EXTENSIONS OF THE BASIC VARIATION FORMULAS*

To extend the usefulness of problems in variation, we can add exponents and square roots to our direct and inverse variation formulas. For example, a fact of physics is that the kinetic energy of an object (energy of motion) varies directly as the square of its velocity, which can be written $E = kv^2$.

We can also combine direct and inverse variation into a single formula. For example, a chemistry principle states, "The volume of a gas is directly proportional to its temperature and inversely proportional to its pressure." This is summarized by the formula $V = \dfrac{kT}{P}$.

EXAMPLE 1: **Translate each variation statement into an algebraic formula, using *k* as the constant of variation:**

A. z varies directly as the cube of T. $z = kT^3$

B. R is inversely proportional to the square root of V. $R = \dfrac{k}{\sqrt{V}}$

C. P varies directly as the square of Q, and inversely as the square root of R. $P = \dfrac{kQ^2}{\sqrt{R}}$

D. B varies directly as the product of the square root of A and the cube of C. $B = k\sqrt{A}\,C^3$

E. y varies directly as the product of x and z, and inversely as the fourth power of w. $y = \dfrac{kxz}{w^4}$

Homework

3. Translate each variation statement into an algebraic formula, using ***k*** as the constant of proportionality:

a. b varies inversely as the 6th power of t.

b. n varies directly as the square root of b.

c. p is inversely proportional to the square of s.

d. v is directly proportional to c.

e. p varies inversely as the cube of x.

f. u is directly proportional to the square of t.

g. t varies directly as the square of h.

h. L is inversely proportional to the cube of s.

i. w varies directly as the 9th power of u.

j. R varies directly as the product of w and y.

k. g is directly proportional to the cube of c.

l. h varies directly as c and inversely as y.

m. x varies directly as the cube of t and inversely as the cube of h.

n. u is directly proportional to the product of c and the cube of r and inversely proportional to m.

o. y varies directly as the square of v and inversely as the cube of d.

p. n varies directly as the product of t and the square of d and inversely as b.

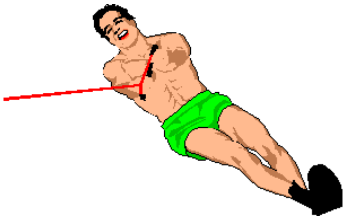
q. m is directly proportional to w and inversely proportional to z.

r. A varies directly as the product of u and the square of x and inversely as p.

s. n is directly proportional to r and inversely proportional to v.

t. c varies directly as the cube of u and inversely as the cube of z.

APPLICATIONS OF VARIATION



EXAMPLE 2: **The number of bathing suits sold is directly proportional to the outside temperature and inversely proportional to the selling price. The number of suits sold is 1,600 when the temperature is 80° and the selling price is $50. Find the number of suits sold when the temperature rises to 95° and the price is reduced to $40.**

Solution: We'll start by letting

B = bathing suits sold
T = temperature
P = price

The first sentence of the problem, “The number of bathing suits sold is directly proportional to the outside temperature, and inversely proportional to the selling price,” gives us our variation formula:

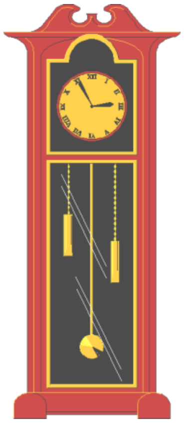
$$B = \frac{kT}{P}$$

The second sentence, “The number of suits sold is 1,600 when the temperature is 80° and the selling price is \$50,” allows us to find the value of k:

$$\begin{aligned} & 1{,}600 = \frac{k(80)}{50} && \text{(substitute the given values)} \\ \Rightarrow\quad & 80k = 80{,}000 && \text{(multiply each side by 50)} \\ \Rightarrow\quad & \boldsymbol{k = 1{,}000} && \text{(divide each side by 80)} \end{aligned}$$

The third sentence, “Find the number of suits sold when the temperature rises to 95° and the price is reduced to \$40,” (with the value $k = 1{,}000$ we just calculated) gives us all the parts needed to compute the number of bathing suits sold under the new set of conditions:

$$\begin{aligned} & B = \frac{kT}{P} && \text{(our variation formula)} \\ \Rightarrow\quad & B = \frac{1{,}000T}{P} && \text{(the constant } \boldsymbol{k} \text{ is 1,000)} \\ \Rightarrow\quad & B = \frac{1{,}000(95)}{40} && \text{(use the new values of } T \text{ and } P\text{)} \\ \Rightarrow\quad & B = \frac{95{,}000}{40} \\ \Rightarrow\quad & \boxed{B = 2{,}375 \text{ bathing suits}} \end{aligned}$$



EXAMPLE 3: **The period of the pendulum (the time for one full swing) in a grandfather clock varies directly as the square root of its length. If the period is 50π when the length is 25, find the period when the length is 49.**

Solution: The first sentence gives us our variation formula:

$$P = k\sqrt{L}$$

Substituting a period of 50π and a length of 25 gives:

$$50\pi = k\sqrt{25}$$

$\Rightarrow \quad 50\pi = 5k \qquad$ (the positive square root of 25 is 5)

$\Rightarrow \quad \mathbf{k = 10\pi} \qquad$ (divide each side by 5)

Now we rewrite our variation formula using the k we just found:

$$P = k\sqrt{L} \Rightarrow P = 10\pi\sqrt{L}$$

Finally, find the period when the length is 49:

$$P = 10\pi\sqrt{49}$$

$$\Rightarrow \quad P = 10\pi(7)$$

$$\Rightarrow \quad \boxed{P = 70\pi}$$

Homework

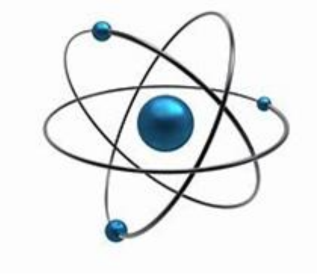
4. The area of a rectangle is directly proportional to its length. If the area is 247 when the length is 19, find the area when the length is 11.

In a variation problem, DON'T forget "***k***", the *constant of variation*.

5. The circumference of a circle is directly proportional to its radius. If the circumference is 40π when the radius is 20, then what is the circumference when the radius is 8?

6. The current in a circuit varies directly as the voltage. If the current is 336 when the voltage is 21, find the current when the voltage is 35.

7. The area of a circle varies directly as the square of its radius. If the area is 9π when the radius is 3), what is the area when the radius is 6?



8. The number of electrons varies directly as the square of the energy level. If the number of electrons is 32 when the energy level is 4, how many electrons are there when the energy level is 3?

9. The energy density is directly proportional to the fourth power of the temperature. If the energy is 512 when the temperature is 4, what is the energy when the temperature is 5?

10. The density of an object varies inversely as the object's volume. If the density is 22 when the volume is 18, what is the density when the volume is 6?

11. The acceleration of an object is inversely proportional to the object's mass. If the acceleration is 9 when the mass is 19, what is the acceleration when the mass is 1?

12. The force of gravity between two objects varies inversely as the square of the distance between the objects. If the force is 4 when the distance is 8, what is the force when the distance is 1?

13. The period of a pendulum varies directly as the square root of its length. If the period is 22π when the length is 121, what is the period when the length is 49?

14. The velocity of an object is directly proportional to the square root of its kinetic energy. If the velocity is 524 when the kinetic energy is 4, what is the velocity when the kinetic energy is 25?

15. The potential energy of an object varies directly as the product of its mass and height. If the potential energy is 80 when the mass

is 4 and the height is 2, find the potential energy if the mass is 14 and the height is 13.

16. The fluid force on an object is directly proportional to the product of its area and its depth. If the fluid force is 1920 when the area is 10 and the depth is 24, find the fluid force if the area is 23 and the depth is 5.

17. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 12 when the temperature is 9 and the pressure is 12, find the volume when the temperature is 5 and the pressure is 16.

18. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 13 when the charge is 13 and the area is 9, find the electric field when the charge is 15 and the area is 3.

Review Problems

19. Translate the variation statement into an algebraic formula, using k as the constant of variation: "E varies directly as the product of x and the cube of y, and inversely as the square root of z."

20. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 80 when the temperature is 40 and the pressure is 5, find the volume when the temperature is 30 and the pressure is 6.

21. The current in a circuit varies directly as the voltage. If the current is 336 when the voltage is 21, find the current when the voltage is 35.

22. The acceleration of an object is inversely proportional to the object's mass. If the acceleration is 9 when the mass is 19, then what is the acceleration when the mass is 1?

23. The potential energy of an object varies directly as the product of its mass and its height. If the potential energy is 462 when the mass is 6 and the height is 7, find the potential energy if the mass is 24 and the height is 17.

24. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 10 when the charge is 15 and the area is 3, find the electric field when the charge is 12 and the area is 6.

25. The kinetic energy of a particle varies directly as the product of its mass and the square of its velocity. Assume that a particle of mass 10 and traveling at a velocity of 8 has a kinetic energy of 320. Find the kinetic energy of a particle with mass 7 and velocity 10.

Solutions

---

1. a. 480 b. 960 c. It increased d. 36 e. It decreased

2. a. 20 b. 5 c. It decreased d. 40 e. It increased

3. a. $b = \frac{k}{t^6}$ b. $n = k\sqrt{b}$ c. $p = \frac{k}{s^2}$ d. $v = kc$

e. $p = \frac{k}{x^3}$ f. $u = kt^2$ g. $t = kh^2$ h. $L = \frac{k}{s^3}$

i. $w = ku^9$ j. $R = kwy$ k. $g = kc^3$ l. $h = \frac{kc}{y}$

m. $x = \frac{kt^3}{h^3}$ n. $u = \frac{kcr^3}{m}$ o. $y = \frac{kv^2}{d^3}$ p. $n = \frac{ktd^2}{b}$

q. $m = \frac{kw}{z}$ r. $A = \frac{kux^2}{p}$ s. $n = \frac{kr}{v}$ t. $c = \frac{ku^3}{z^3}$

4. 143 **5**. 16π **6**. 560 **7**. 36π

8. 18 **9**. 1,250 **10**. 66 **11**. 171

12. 256 **13**. 14π **14**. 1,310 **15**. 1,820

16. 920 **17**. 5 **18**. 45 **19**. $E = \frac{ky^3}{\sqrt{z}}$

20. 50

21. In electronics, current is denoted by the letter i.

$i = kV \Rightarrow 336 = k \cdot 21 \Rightarrow k = 16 \Rightarrow i = 16V$,
so when $V = 35$, $i = 16 \cdot 35 = 560$.

22. $a = 171$

23. $E = kmh \Rightarrow 462 = k \cdot 6 \cdot 7 \Rightarrow k = 11 \Rightarrow E = 11mh$
$\Rightarrow E = 11(24)(17) = 4488$

24. $F = \frac{kC}{A}$; $k = 2$; $F = 4$

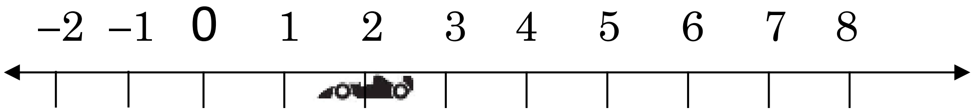
25. $E = kmv^2$ $E = 350$
[In physics, you'll learn that $k = \frac{1}{2}$.]

"It is never too late to be what you might have been."

- George Eliot

CH XX – AVERAGE VELOCITY

Now we look at a number line – a simple horizontal axis – and talk about some object moving left or right on the line.



It could be a racecar speeding down the backstretch or a subatomic proton hurtling through a linear accelerator.

NOTATION

Throughout this chapter, we'll let ***t*** represent ***time*** and ***s*** represent the *position* of the object on the line. We agree that $s > 0$ when the object is to the right of 0, and $s < 0$ on the left side of 0. It is also important that we consider the position, s, to be a function of time, t. In other words, as the time changes, the position of the object may also change. We can even write the position as $\boldsymbol{s(t)}$ [read: "***s*** of ***t***"] to express the view that the position, ***s,*** is a function of the time, ***t***.

In addition, to emphasize the math, we won't worry about units. But for your edification, if position is measured in meters from the origin (as either a positive number, negative number, or zero) and time is measured in seconds (common in physics), then the

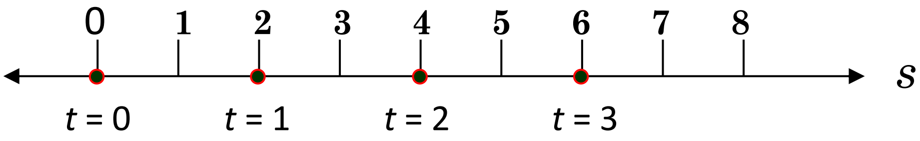
Speed v. Velocity

65 mph is *speed*.
65 mph To The Right is *velocity*.

unit of **velocity** is *meters per second*, or m/s, or $\frac{\text{m}}{\text{s}}$, followed by the direction "left" or "right."

☐ TIME AND POSITION

Consider the following real line (which we could call the s-axis). Each dot on the line represents the **position** of an object at the moment of **time** specified, where t might be measured in seconds:



Here's what this little picture means. At the "beginning," when $t = 0$, the position is $s = 0$. One second later, when $t = 1$, the object is at $s = 2$. One second after that, when $t = 2$, s has the value 4. When $t = 3$, the object is at 6. Expressing these facts in function notation, we write:

$$s(0) = 0 \qquad s(1) = 2 \qquad s(2) = 4 \qquad s(3) = 6$$

[The notation ***s*(*t*)** is read "*s* of *t*."]

We can also express this number line picture with a table of values:

t	0	1	2	3
s	0	2	4	6

Our goal now is to predict where the object will be at some time in the **future.** For example, when $t = 10$, that is, 10 seconds after the object begins moving, where will it be? To do this, we ask the question: Given the data in the table, can we find a formula for the position of the object as a function of time? Looking either at the real line (the s-axis) or the table, it seems (at least from the data we have) that the position is always twice the value of time: ***s* is twice *t***. So we can write the formula

$$s(t) = 2t \qquad \text{[The position at time } t \text{ is twice the } t\text{-value]}$$

Thus, to predict the position s when $t = 10$, all we have to do is calculate $s(\mathbf{10})$, which is $2 \cdot \mathbf{10} = 20$. Therefore, after 10 seconds have passed, the object is at **position 20**. Notice that the object in this example never has a negative value of s, and that it moves to the right as the value of time increases.

Homework

1. The following table represents the position of a car at various moments in time. Predict the position of the car when $t = 850$.

t	1	2	3	4	5
s	4	7	12	19	28

❑ *AVERAGE VELOCITY*

Have you studied **delta** notation? A change in y, which might be written as the difference $y_2 - y_1$, can be expressed as Δy. In discussing velocity, we need the notion of a change in time, denoted by Δt, and as a consequence, a change in position, written Δs.

The symbol Δ is the Greek capital letter delta, and is a common symbol representing a **change** in something.

Now we're ready to define the concept of ***average velocity***:

average velocity = the change in position divided by the change in time.

It's a measure of the *net distance* traveled in a specified amount of time. We use the symbol $\bar{v}$ (v-bar) to represent average velocity:

$$\boxed{\bar{v} = \frac{\Delta s}{\Delta t}}$$

This formula is very similar to the formula involving distance, rate, and time:

$$r = \frac{d}{t}$$

Four Notes:

1. The change in position, Δs, does <u>not</u> take into account the actual distance traveled by the object. Suppose you and I are standing on the s-axis at the same spot, say $s = 10$. You run many miles to the left and then turn around and run back, ending at $s = 12$. Your change in position is 12 – 10 = **2**. I, being the lazy bum that I am, simply stroll directly from $s = 10$ to $s = 12$. My change in position is 12 – 10 = **2**, the same change in position as yours. We started at the same spot and we ended at the same spot, and so we each experienced the same change in position — our Δs's are the same (even though you ran a much farther distance than I did.)

2. The formula for average velocity may remind you of the definition of **slope**, $\frac{\Delta y}{\Delta x}$. This is no coincidence. Each of the concepts — *slope* and *average velocity* — is described by a ratio of changes, and each of them forms the foundation for the branch of math called *calculus*.

3. Just as the slope of a line can be negative, so too can the average velocity be negative. You'll see this in Example 2 below. By the way, this is one of the reasons that we refer to the theme of this chapter using the term *velocity* rather than speed.

4. You might notice at this point that your calculation for average velocity will involve two subtractions followed by a division. The order in which you perform the subtractions is up to you, but you MUST be consistent in that order. Nevertheless, tradition dictates that we subtract initial values from final values. For example, if an object is at $s = 3$ and then later is at $s = 10$, we agree to calculate 10 – 3 rather than 3 – 10. So our average velocity formula looks like

Velocity has direction; speed does not.

$$\boxed{\bar{v} = \frac{\textit{final position} - \textit{initial position}}{\textit{final time} - \textit{initial time}}}$$

EXAMPLE 1: **A bird takes off from position $s = 3$ when $t = 5$ and lands at position $s = 15$ when $t = 9$. Find the average velocity for the time interval $t = 5$ to 9.**

Solution: The average velocity is the change in position (the difference of the positions) divided by the change in time (the difference of the times):

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{15-3}{9-5} = \frac{12}{4} = \boxed{3}$$

EXAMPLE 2: **A racecar is at position $s = 8$ when $t = 0$ and then drives to position $s = -10$ in 2 seconds. Find the average velocity during the 2-second interval.**

Solution: The racecar started when $t = 0$ and ended when $t = 2$. Using the formula for average velocity, we calculate

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{-10-8}{2-0} = \frac{-18}{2} = \boxed{-9}$$

Why is the velocity negative?

Homework

2. A snail is at position $s = 22$ at $t = 4$ and moves to position $s = 50$ at $t = 18$. Find the average velocity, $\bar{v}$, for the time interval $t = 4$ to 18.

3. A rocket is at position $s = 1{,}000$ at $t = 20$ and rises to position $s = 5{,}500$ at $t = 65$. Find the average velocity, $\bar{v}$.

EXAMPLE 3: **Let $s(t) = t^2$. Calculate each of the following:**

a. $s(0)$ b. $s(20)$ c. $\bar{v}$ from $t = 5$ to $t = 15$

Solution:

a. $s(0)$ means let $t = 0$ and calculate s using the given formula:

$$s(0) = 0^2 = \mathbf{0}$$

b. s(20) is the position when $t = 20$:

$$s(20) = 20^2 = \mathbf{400}$$

t	s
0	0
5	25
15	225
20	400

c. i) $s(5) = 5^2 = 25$

ii) $s(15) = 15^2 = 225$

The average velocity, $\bar{v}$, is calculated as follows:

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{225-25}{15-5} = \frac{200}{10} = \mathbf{20}$$

Homework

For each position function, calculate the average velocity, $\bar{v}$, during the given time interval:

4. $s(t) = 3t + 7$ $t = 3$ to $t = 10$

5. $s(t) = t^2 + 3t + 7$ $t = 1$ to $t = 5$

6. $s(t) = t^3$ $t = 3$ to $t = 5$

7. $s(t) = t^3 + t^2 + 3t + 7$ $t = 1$ to $t = 2$

8. $s(t) = \frac{1}{t}$ $t = 4$ to $t = 6$

9. $s(t) = \sqrt{t}$ $t = 9$ to $t = 36$

10. $s(t) = \frac{1}{\sqrt{t}}$ $t = 4$ to $t = 100$

11. $s(t) = 7$ $t = \pi$ to $t = 50$

12. For each position function, calculate the average velocity, $\bar{v}$, during the time interval from $t = 0$ to $t = 1$:

    a. $s(t) = at + b$

    b. $s(t) = at^2 + bt + c$

    c. $s(t) = at^3 + bt^2 + ct + d$

Solutions

1. 722,503

2. 2

3. 100

4. 3

5. 9

6. 49

7. 13

8. –0.04

9. 0.11

10. –0.004

11. 0

12. I want to see what you come up with.

“STRIVE FOR *progress,*
NOT *perfection.”*